Thin II_1 factors with no Cartan subalgebras

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Abstract

It is a wide open problem to give an intrinsic criterion for a II_1 factor M to admit a Cartan subalgebra A. When $A \subset M$ is a Cartan subalgebra, the A-bimodule $L^2(M)$ is "simple" in the sense that the left and right action of A generate a maximal abelian subalgebra of $B(L^2(M))$. A II_1 factor M that admits such a subalgebra A is said to be s-thin. Very recently, Popa discovered an intrinsic local criterion for a II_1 factor M to be s-thin and left open the question whether all s-thin II_1 factors admit a Cartan subalgebra. We answer this question negatively by constructing s-thin II_1 factors without Cartan subalgebras.

1 Introduction

One of the main decomposability properties of a II_1 factor M is the existence of a *Cartan subalgebra* $A \subset M$, i.e. a maximal abelian subalgebra (MASA) whose normalizer $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates M as a von Neumann algebra. Indeed by [FM75], when M admits a Cartan subalgebra, then M can be realized as the von Neumann algebra $L_\Omega(\mathcal{R})$ associated with a countable equivalence relation $\mathcal{R}$, possibly twisted by a scalar 2-cocycle Ω. If moreover this Cartan subalgebra is unique in the appropriate sense, this decomposition $M = L_\Omega(\mathcal{R})$ is canonical.

Although a lot of progress on the existence and uniqueness of Cartan subalgebras has been made (see e.g. [OP07, PV11]), there is so far no intrinsic local criterion to check whether a given II_1 factor admits a Cartan subalgebra. When $A \subset M$ is a Cartan subalgebra, then $A \subset M$ is in particular an *s-MASA*, meaning that the A-bimodule ${}_A L^2(M)_A$ is *cyclic*, i.e. there exists a vector $\xi \in L^2(M)$ such that $A\xi A$ spans a dense subspace of $L^2(M)$. Although it was already shown in [Pu59] that the hyperfinite II_1 factor R admits an s-MASA $A \subset R$ that is *singular* (i.e. that satisfies $\mathcal{N}_R(A)'' = A$), all examples of s-MASAs so far were inside II_1 factors that also admit a Cartan subalgebra.

Very recently in [Po16], Popa discovered that the existence of an s-MASA in a II_1 factor M is an intrinsic local property. He proved that a II_1 factor M admits an s-MASA if and only if M satisfies the s-thin approximation property: for every finite partition of the identity $p_1, \ldots, p_n$ in M, every finite subset $\mathcal{F} \subset M$ and every $\varepsilon > 0$, there exists a finer partition of the identity $q_1, \ldots, q_m$ and a single vector $\xi \in L^2(M)$ such that every element in $\mathcal{F}$ can be approximated up to ϵ in $\|\cdot\|_2$ by linear combinations of the $q_i \xi q_j$.

Although an s-MASA can be singular and although it is even proved in [Po16, Corollary 4.2] that every s-thin II_1 factor admits uncountably many non conjugate singular s-MASAs, as said above, all known s-thin factors so far also admit a Cartan subalgebra and Popa poses as [Po16, Problem 5.1.2] to give examples of s-thin factors without Cartan subalgebras. We solve this problem here by constructing s-thin II_1 factors M that are even *strongly solid:* whenever $B \subset M$ is a diffuse amenable von Neumann subalgebra, the normalizer $\mathcal{N}_M(B)''$ stays amenable. Clearly, nonamenable strongly solid II_1 factors have no Cartan subalgebras.

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We obtain this new class of strongly solid II_1 factors by applying Popa's deformation/rigidity theory to Shlyakhtenko's A-valued semicircular systems (see [Sh97] and Section 3 below). When A is abelian, this provides a rich source of examples of MASAs with special properties, like MASAs satisfying the s-thin approximation property of [Po16].

Generalizing Voiculescu's free Gaussian functor [Vo83], the data of Shlyakhtenko's construction consists of a tracial von Neumann algebra (A, τ) and a symmetric A-bimodule ${}_A H_A$, where the symmetry is given by an anti-unitary operator $J : H \to H$ satisfying $J^2 = 1$ and $J(a \cdot \xi \cdot b) = b^* \cdot J\xi \cdot a^*$. The construction produces a tracial von Neumann algebra M containing A such that ${}_A L^2(M)_A$ can be identified with the full Fock space

$$L^2(A) \oplus \bigoplus_{n \geq 1} \big(\underbrace{H \otimes_A \cdots \otimes_A H}_{n \text{ times}}\big) .$$

In the same way as the free Gaussian functor transforms direct sums of real Hilbert spaces into free products of von Neumann algebras, the construction of [Sh97] transforms direct sums of A-bimodules into free products that are amalgamated over A. Therefore, the deformation/rigidity results and methods for amalgamated free products introduced in [IPP05, Io12], and in particular Popa's s-malleable deformation obtained by "doubling and rotating" the A-bimodule, can be applied and yield the following result, proved in Corollaries 4.2 and 6.2 below (see Theorem 6.1 for the most general statement).

Theorem A. *Let (A, τ) be a tracial von Neumann algebra and let M be the von Neumann algebra associated with a symmetric A-bimodule ${}_A H_A$. Assume that ${}_A H_A$ is weakly mixing (Definition 2.2) and that the left action of A on H is faithful. Then, M has no Cartan subalgebra. If moreover ${}_A H_A$ is mixing and A is amenable, then M is strongly solid.*

In the particular case where A is diffuse abelian and the bimodule ${}_A H_A$ is weakly mixing, we get that $A \subset M$ is a singular MASA. Very interesting examples arise as follows by taking $A = L^\infty(K, \mu)$ where K is a second countable compact group with Haar probability measure μ. Whenever ν is a probability measure on K, we consider the A-bimodule H_ν given by

$$H_\nu = L^2(K \times K, \mu \times \nu) \quad \text{with} \;\; (F \cdot \xi \cdot G)(x, y) = F(xy)\, \xi(x, y)\, G(x) \, , \tag{1.1}$$

for all $F, G \in A$ and $\xi \in H_\nu$. We assume that ν is symmetric and use the symmetry

$$J_\nu : H_\nu \to H_\nu : (J\xi)(x, y) = \overline{\xi(xy, y^{-1})} \quad \text{for all} \;\; x, y \in K \, . \tag{1.2}$$

We denote by M the tracial von Neumann algebra associated with the A-bimodule (H_ν, J_ν).

The A-bimodule H_ν is weakly mixing if and only if the measure ν has no atoms, while H_ν is mixing when the probability measure ν is c_0, meaning that the convolution operator $\lambda(\nu)$ on $L^2(K)$ is compact (see Definition 7.2 and Proposition 7.3). So for all c_0 probability measures ν on K, we get that M is strongly solid.

On the other hand, when the measure ν is concentrated on a subset of the form $F \cup F^{-1}$, where $F \subset K$ is *free* in the sense that every reduced word with letters from $F \cup F^{-1}$ defines a nontrivial element of K, then $A \subset M$ is an s-MASA.

In Theorem 7.5, we construct a compact group K, a free subset $F \subset K$ generating K and a symmetric c_0 probability measure ν with support $F \cup F^{-1}$. For this, we use results of [AR92, GHSSV07] on the spectral gap and girth of a random Cayley graph of the finite group $\mathrm{PGL}(2, \mathbb{Z}/p\mathbb{Z})$. As a consequence, we obtain the first examples of s-thin II_1 factors that have no Cartan subalgebra, solving [Po16, Problem 5.1.2], which was the motivation for our work.

Theorem B. *Taking a compact group K and a symmetric probability measure ν on K as above, the associated II$_1$ factor M is nonamenable, strongly solid and the canonical subalgebra $A \subset M$ is an s-MASA.*

As we explain in Remark 3.5, the so-called free Bogoljubov crossed products $L(\mathbb{F}_\infty) \rtimes G$ associated with an (infinite dimensional) orthogonal representation of a countable group G can be written as the von Neumann algebra associated with a symmetric A-bimodule where $A = L(G)$. Therefore, our Theorem A is a generalization of similar results proved in [Ho12b] for free Bogoljubov crossed products. Although free Bogoljubov crossed products $M = L(\mathbb{F}_\infty) \rtimes G$ with G abelian provide examples of MASAs $L(G) \subset M$ with interesting properties (see [HS09, Ho12a]), $L(G) \subset M$ can never be an s-MASA (see Remark 7.4).

The point of view of A-valued semicircular systems is more flexible and even offers advantages in the study of free Bogoljubov crossed products $M = L(\mathbb{F}_\infty) \rtimes G$. Indeed, in Corollary 6.4, we prove that these II$_1$ factors M never have a Cartan subalgebra, while in [Ho12b], this could only be proved for special classes of orthogonal representations.

In Theorem 5.1, we prove several maximal amenability results for the inclusion $A \subset M$ associated with a symmetric A-bimodule (H, J), by combining the methods of [Po83, BH16]. Again, these results generalize [Ho12a, Ho12b] where the same was proved for free Bogoljubov crossed products.

We finally make some concluding remarks on the existence of c_0 probability measures supported on free subsets of a compact group. On an *abelian* compact group K, a probability measure ν is c_0 if and only if its Fourier transform $\widehat{\nu}$ tends to zero at infinity as a function from $\widehat{K}$ to $\mathbb{C}$. Of course, no two elements of an abelian group are free, but the abelian variant of being free is the so-called independence property: a subset F of an abelian compact group K is called independent if any linear combination of distinct elements in F with coefficients in $\mathbb{Z} \setminus \{0\}$ defines a non zero element in K. It was proved in [Ru60] that there exist closed independent subsets of the circle group $\mathbb{T}$ that carry a c_0 probability measure. It would be very interesting to get a better understanding of which, necessarily non abelian, compact groups admit c_0 probability measures supported on a free subset and we conjecture that these exist on the groups $\mathrm{SO}(n)$, $n \geq 3$.

2 Preliminaries

Let (A, τ) be a tracial von Neumann algebra.

Definition 2.1. A *symmetric* A-bimodule (H, J) is an A-bimodule ${}_A H_A$ equipped with an anti-unitary operator $J \colon H \to H$ such that $J^2 = 1$ and

$$J(a \cdot \xi \cdot b) = b^* \cdot J\xi \cdot a^*, \qquad \forall a, b \in A \, .$$

A vector ξ in a right (resp. left) A-module H is said to be right (resp. left) bounded if there exists a $\kappa > 0$ such that $\|\xi a\| \leq \kappa \|a\|_2$ (resp. $\|a\xi\| \leq \kappa \|a\|_2$) for all $a \in A$. Whenever ξ is right bounded, we denote by $\ell(\xi)$ the map $L^2(A) \to H : a \mapsto \xi a$. Similarly, when ξ is left bounded, we denote by $r(\xi)$ the map $L^2(A) \to H : a \mapsto a\xi$.

Given right bounded vectors ξ, η, the operator $\ell(\xi)^* \ell(\eta)$ belongs to A and is denoted $\langle \xi, \eta \rangle_A$. This defines an A-valued scalar product associated with the right A-module H. Similarly, if $\xi, \eta \in H$ are left bounded vectors, we define an A-valued scalar product associated with the left A-module H by ${}_A\langle \xi, \eta \rangle = J r(\xi)^* r(\eta) J \in A$. Here, J denotes the canonical involution on $L^2(A)$.

Popa's non intertwinability condition (see [Po03, Section 2]) saying that $B \not\prec_M A$ is equivalent with the existence of a sequence of unitaries $b_n \in \mathcal{U}(B)$ such that $\lim_n \|E_A(xb_ny)\|_2 = 0$ for all $x, y \in M$ can be viewed as a weak mixing condition for the B-A-bimodule ${}_BL^2(M)_A$ (cf. the notions of relative (weak) mixing in [Po05, Definition 2.9]). This then naturally lead to the notion of a mixing, resp. weakly mixing bimodule in [PS12].

Definition 2.2 ([PS12])**.** Let (A, τ) and (B, τ) be tracial von Neumann algebras and ${}_BH_A$ a B-A-bimodule.

1. ${}_BH_A$ is called *left weakly mixing* if there exists a net of unitaries $b_n \in \mathcal{U}(B)$ such that for all right bounded vectors $\xi, \eta \in H$, we have
$$\lim_n \|\langle b_n\xi, \eta\rangle_A\|_2 = 0 \; .$$

2. ${}_BH_A$ is called *left mixing* if every net $b_n \in \mathcal{U}(B)$ tending to 0 weakly satisfies
$$\lim_n \|\langle b_n\xi, \eta\rangle_A\|_2 = 0$$
for all right bounded vectors $\xi, \eta \in H$.

We similarly define the notions of *right (weak) mixing*. When ${}_AH_A$ is a symmetric A-bimodule, left (weak) mixing is equivalent with right (weak) mixing and we simply refer to these properties as (weak) mixing.

In [Po03, Section 2], Popa proved that the intertwining relation $B \prec_M A$ is equivalent with the existence of a nonzero B-A-subbimodule of $L^2(M)$ having finite right A-dimension. In the same way, one gets the following characterization of weakly mixing bimodules. For details, see [PS12] and [Bo14, Theorem A.2.2].

Proposition 2.3 ([Po03, PS12, Bo14])**.** *Let (A, τ) and (B, τ) be tracial von Neumann algebras and ${}_BH_A$ a B-A-bimodule. The following are equivalent:*

1. ${}_BH_A$ is left weakly mixing;

2. $\{0\}$ is the only B-A-subbimodule of ${}_BH_A$ of finite A-dimension;

3. ${}_B(H \otimes_A \overline{H})_B$ has no nonzero B-central vectors.

3 Shlyakhtenko's A-valued semicircular systems

We first recall Voiculescu's free Gaussian functor from the category of real Hilbert spaces to the category of tracial von Neumann algebras. Let $H_\mathbb{R}$ be a real Hilbert space and let H be its complexification. The *full Fock space* of H is defined as
$$\mathcal{F}(H) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n} \; .$$
The unit vector Ω is called the *vacuum vector*. Given a vector $\xi \in H$, we define the *left creation operator* $\ell(\xi) \in B(\mathcal{F}(H))$ by
$$\ell(\xi)(\Omega) = \xi \quad \text{and} \quad \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n \; .$$
Put
$$\Gamma(H_\mathbb{R})'' := \{\ell(\xi) + \ell(\xi)^* \mid \xi \in H_\mathbb{R}\}'' \; .$$

This von Neumann algebra is equipped with the faithful trace given by $\tau(\cdot) = \langle \cdot\, \Omega, \Omega\rangle$. In [Vo83], it is proved that the operator $\ell(\xi) + \ell(\xi)^*$ has a semicircular distribution with respect to the trace τ and that $\Gamma(H_\mathbb{R})'' \cong L(\mathbb{F}_{\dim H_\mathbb{R}})$. By the functoriality of the construction, any orthogonal transformation u of $H_\mathbb{R}$ gives rise to an automorphism α_u of $\Gamma(H_\mathbb{R})''$ satisfying $\alpha_u(\ell(\xi) + \ell(\xi)^*) = \ell(u\xi) + \ell(u\xi)^*$ for all $\xi \in H_\mathbb{R}$. So, every orthogonal representation $\pi : G \to O(H_\mathbb{R})$ of a countable group G gives rise to the *free Bogoljubov action* $\sigma_\pi : G \curvearrowright \Gamma(H_\mathbb{R})''$ given by $\sigma_\pi(g) = \alpha_{\pi(g)}$ for all $g \in G$.

In [Sh97], Shlyakhtenko introduced a generalization of Voiculescu's free Gaussian functor, this time being a functor from the category of symmetric A-bimodules (where A is any von Neumann algebra) to the category of von Neumann algebras containing A. We will here repeat this construction in the case where A is a tracial von Neumann algebra.

Let (A, τ) be a tracial von Neumann algebra and let (H, J) be a symmetric A-bimodule. We denote by $H^{\otimes_A^n}$ the n-fold Connes tensor product $H \otimes_A H \otimes_A \cdots \otimes_A H$. The full Fock space of the A-bimodule ${}_AH_A$ is defined by

$$\mathcal{F}_A(H) = L^2(A) \oplus \bigoplus_{n=1}^{\infty} H^{\otimes_A^n} \; . \tag{3.1}$$

We denote by $\mathcal{H}$ the set of left and right A-bounded vectors in H. Since A is a tracial von Neumann algebra, $\mathcal{H}$ is dense in H. Given a right bounded vector $\xi \in H$, we define the left creation operator $\ell(\xi)$ analogous to the case where $A = \mathbb{C}$ by

$$\begin{aligned} &\ell(\xi)(a) = \xi a, \quad a \in A \; , \\ &\ell(\xi)(\xi_1 \otimes_A \ldots \otimes_A \xi_n) = \xi \otimes_A \xi_1 \otimes_A \ldots \otimes_A \xi_n, \quad \xi_i \in \mathcal{H} \; . \end{aligned}$$

Note that $a\ell(\xi) = \ell(a\xi)$ and $\ell(\xi)a = \ell(\xi a)$ for $a \in A$ and that the adjoint map $\ell(\xi)^*$ satisfies

$$\begin{aligned} &\ell(\xi)^*(a) = 0 \quad \text{for all } a \in L^2(A) \; , \\ &\ell(\xi)^*(\xi_1 \otimes_A \ldots \otimes_A \xi_n) = \langle \xi, \xi_1\rangle_A \xi_2 \otimes_A \ldots \otimes_A \xi_n \quad \text{for } \xi_i \in \mathcal{H} \; . \end{aligned}$$

Definition 3.1. Given a tracial von Neumann algebra (A, τ) and a symmetric A-bimodule (H, J), we consider the full Fock space $\mathcal{F}_A(H)$ given by (3.1) and define

$$\Gamma(H, J, A, \tau)'' := A \vee \{\ell(\xi) + \ell(\xi)^* \mid \xi \in \mathcal{H},\ J\xi = \xi\}'' \subset B(\mathcal{F}_A(H)) \; ,$$

where $A \subset B(\mathcal{F}_A(H))$ is given by the left action on $\mathcal{F}_A(H)$. We also have

$$\Gamma(H, J, A, \tau)'' = A \vee \{\ell(\xi) + \ell(J\xi)^* \mid \xi \in \mathcal{H}\}'' \; .$$

We denote by Ω the vacuum vector in $\mathcal{F}_A(H)$ given by $\Omega = 1_A \in L^2(A)$. We define τ as the vector state on $M = \Gamma(H, J, A, \tau)''$ given by the vacuum vector Ω. Whenever $n \geq 1$ and $\xi_1, \ldots, \xi_n \in \mathcal{H}$, we define the Wick product as in [HR10, Lemma 3.2] by

$$W(\xi_1, \ldots, \xi_n) = \sum_{i=0}^{n} \ell(\xi_1) \cdots \ell(\xi_i) \ell(J\xi_{i+1})^* \cdots \ell(J\xi_n)^* \; . \tag{3.2}$$

As in [HR10, Lemma 3.2], we get that $W(\xi_1, \ldots, \xi_n) \in M$ and

$$W(\xi_1, \ldots, \xi_n)\Omega = \xi_1 \otimes_A \cdots \otimes_A \xi_n \; .$$

These elements, with $n \geq 1$, span a $\|\cdot\|_2$-dense subspace of $M \ominus A$. Together with A, they span a $\|\cdot\|_2$-dense $*$-subalgebra of M.

Proposition 3.2 ([Sh97]). *The state $\tau(\cdot) = \langle \cdot\, \Omega, \Omega\rangle$ defined above is a faithful trace on M.*

Proof. Define $\mathcal{J} \colon \mathcal{F}_A(H) \to \mathcal{F}_A(H)$ by $\mathcal{J}(a) = a^*$ for $a \in A$ and

$$\mathcal{J}(\xi_1 \otimes_A \cdots \otimes_A \xi_n) = J\xi_n \otimes_A \cdots \otimes_A J\xi_1$$

for $\xi_1, \ldots, \xi_n \in \mathcal{H}$. Then $\mathcal{J}$ is an anti-unitary map satisfying $\mathcal{J}^2 = 1$. One easily checks that $\mathcal{J}\ell(\xi)\mathcal{J} = r(J\xi)$ for all $\xi \in \mathcal{H}$ and that $\mathcal{J}a\mathcal{J}$ is just right multiplication by a^* on $\mathcal{F}_A(H)$. This implies that $\mathcal{J}M\mathcal{J}$ commutes with M. Indeed, for $\xi, \eta \in \mathcal{H}$ with $J\xi = \xi$ and $J\eta = \eta$, we have $\langle \xi, a\eta\rangle_A = {}_A\langle \xi a, \eta\rangle$ since

$$\begin{aligned}\langle Jr(\xi a)^* r(\eta) Jx, y\rangle &= \langle r(\xi a)y^*, r(\eta)x^*\rangle = \langle y^*\xi a, x^*\eta\rangle = \langle J(x^*\eta), J(y^*\xi a)\rangle \\ &= \langle \eta x, a^*\xi y\rangle = \langle \ell(\xi)^*\ell(a\eta)x, y\rangle ,\end{aligned}$$

for all $x, y \in A$. It follows that

$$(\ell(\xi)^* r(\eta) + \ell(\xi) r(\eta)^*)(a) = \langle \xi, a\eta\rangle_A = {}_A\langle \xi a, \eta\rangle = (r(\eta)^*\ell(\xi) + r(\eta)\ell(\xi)^*)(a), \quad \forall a \in A .$$

Since $\ell(\xi)$ and $r(\eta)^*$ clearly commute when restricted to $\mathcal{F}_A(H) \ominus L^2(A)$, it follows that $\ell(\xi) + \ell(\xi)^*$ commutes with $r(\eta) + r(\eta)^*$. We conclude that M commutes with $\mathcal{J}M\mathcal{J}$.

Next, we show that $\mathcal{J}(x\Omega) = x^*\Omega$ for all $x \in M$. This clearly holds for $x \in A$ so it suffices to prove it for x of the form $x = W(\xi_1, \ldots, \xi_n)$ with $\xi_i \in \mathcal{H}$. We have

$$\begin{aligned}\mathcal{J}(W(\xi_1, \ldots, \xi_n)\Omega) &= \mathcal{J}(\xi_1 \otimes_A \cdots \otimes_A \xi_n) = J\xi_n \otimes_A \cdots \otimes_A J\xi_1 \\ &= W(J\xi_n, \ldots, J\xi_1)\Omega = W(\xi_1, \ldots, \xi_n)^*\Omega .\end{aligned}$$

We now get that

$$\begin{aligned}\tau(xy) &= \langle xy\Omega, \Omega\rangle = \langle x\mathcal{J}(y^*\Omega), \Omega\rangle = \langle x\mathcal{J}y^*\mathcal{J}\Omega, \Omega\rangle = \langle \mathcal{J}y^*\mathcal{J}x\Omega, \Omega\rangle \\ &= \langle x\Omega, \mathcal{J}y\mathcal{J}\Omega\rangle = \langle x\Omega, y^*\Omega\rangle = \langle yx\Omega, \Omega\rangle = \tau(yx) ,\end{aligned}$$

for all $x, y \in M$ and hence τ is a trace.

It is easy to check that $\Omega \in \mathcal{F}_A(H)$ is a cyclic vector for both M and $\mathcal{J}M\mathcal{J}$. Hence Ω is also separating for M and it follows that τ is faithful. $\square$

By construction, we have that $L^2(M) \cong \mathcal{F}_A(H)$ as A-bimodules.

In [Sh97], Shlyakhtenko used the terminology *A-valued semicircular system* for the family $\{\ell(\xi) + \ell(\xi)^* \mid \xi \in \mathcal{H}, J\xi = \xi\}$, as an analogue to the free Gaussian functor case, where the operator $\ell(\xi) + \ell(\xi)^*$ has a semicircular distribution with respect to τ.

Example 3.3. 1. When $H = L^2(A)$ is the trivial A-bimodule with $J(a) = a^*$, we simply get

$$\Gamma(H, J, A, \tau)'' = A \mathbin{\overline{\otimes}} L^\infty[0, 1] .$$

   Indeed, A commutes with $\ell(\hat{1}) + \ell(\hat{1})^*$ and they together generate $\Gamma(H, J, A, \tau)''$. In particular, we see that $\Gamma(H, J, A, \tau)''$ is not always a factor.

2. When $H = L^2(A) \otimes L^2(A)$ is the coarse A-bimodule with $J(a \otimes b) = b^* \otimes a^*$, we get

$$\Gamma(H, J, A, \tau)'' = (A, \tau) * L^\infty[0, 1] .$$

This example shows that the construction of $\Gamma(H, J, A, \tau)''$ may depend on the trace on A. Indeed, if $A = \mathbb{C}^2$ we can consider the trace τ_δ for any $\delta \in (0,1)$ given by $\tau_\delta(a,b) = \delta a + (1-\delta)b$, $a, b \in \mathbb{C}$. By [Dy92, Lemma 1.6] we have that $L(\mathbb{Z}) * (A, \tau_\delta) = L(\mathbb{F}_{1+2\delta(1-\delta)})$, the interpolated free group factor. It is wide open whether the interpolated free group factors are all isomorphic. So at least, there is no obvious isomorphism between $\Gamma(H, J, A, \tau_{\delta_1})''$ and $\Gamma(H, J, A, \tau_{\delta_2})''$ for $\delta_1 \neq \delta_2$. In Example 3.6, we shall actually see that even the factoriality of $\Gamma(H, J, A, \tau)''$ may depend on the choice of the trace τ. For a general factoriality criterion for $\Gamma(H, J, A, \tau)''$, see Theorem 6.1.

Note that the construction of $\Gamma(H, J, A, \tau)''$ is functorial in the following sense. If $U \in \mathcal{U}(H)$ is a unitary operator that is A-bimodular and commutes with J, then U defines a trace-preserving automorphism of $M = \Gamma(H, J, A, \tau)''$ in the following way. Since U is A-bimodular, we can define a unitary U^n on $H^{\otimes_A^n}$ by $U^n(\xi_1 \otimes_A \cdots \otimes_A \xi_n) = (U\xi_1 \otimes_A \cdots \otimes_A U\xi_n)$. The direct sum of these unitaries (and the identity on $L^2(A)$) then gives an A-bimodular unitary operator on $\mathcal{F}_A(H)$, which we will still denote by U. Note that $U\ell(\xi)U^* = \ell(U\xi)$ for all $\xi \in \mathcal{H}$. Since U commutes with J, it follows that $UMU^* = M$ so that $\operatorname{Ad} U$ defines an automorphism of M.

Recall that for Voiculescu's free Gaussian functor, we have that the direct sum of Hilbert spaces translates into the free product of von Neumann algebras, in the sense that $\Gamma(H_1 \oplus H_2) = \Gamma(H_1) * \Gamma(H_2)$. In the setting of A-bimodules in general, we instead get the amalgamated free product over A as stated in the following proposition.

Proposition 3.4 ([Sh97, Proposition 2.17])**.** *Let (H_1, J_1) and (H_2, J_2) be symmetric A-bimodules. Put $H = H_1 \oplus H_2$ and $J = J_1 \oplus J_2$. Then*

$$\Gamma(H, J, A, \tau)'' \cong \Gamma(H_1, J_1, A, \tau)'' *_A \Gamma(H_2, J_2, A, \tau)'' ,$$

with respect to the unique trace-preserving conditional expectation onto A.

Remark 3.5. As we recalled in the beginning of this section, to every orthogonal representation $\pi : G \to O(K_\mathbb{R})$ of a countable group G on a real Hilbert space $K_\mathbb{R}$ is associated the free Bogoljubov action $\sigma_\pi : G \curvearrowright \Gamma(K_\mathbb{R})''$. Write $A = L(G)$ and equip A with its canonical tracial state τ. Denote by K the complexification of $K_\mathbb{R}$ and define the symmetric A-bimodule ${}_A H_A$ given by

$$\begin{aligned} H = \ell^2(G) \otimes K \quad &\text{with} \quad u_g \cdot (\delta_h \otimes \xi) \cdot u_k = \delta_{ghk} \otimes \pi(g)\xi \\ &\text{and} \quad J(\delta_h \otimes \xi) = \delta_{h^{-1}} \otimes \pi(h^{-1})\overline{\xi} \end{aligned} \tag{3.3}$$

where $(\delta_g)_{g \in G}$ denotes the canonical orthonormal basis of $\ell^2(G)$. It is now straightforward to check that there is a canonical trace preserving isomorphism

$$\Gamma(H, J, A, \tau)'' \cong \Gamma(K_\mathbb{R})'' \rtimes^{\sigma_\pi} G$$

that maps A onto $L(G)$ identically.

Example 3.6. This final example illustrates that even the factoriality of $\Gamma(H, J, A, \tau)''$ may depend on the choice of τ. Take $A = \mathbb{C}^2$, $\alpha \in \operatorname{Aut}(A)$ the flip automorphism and $H = \mathbb{C}^2$ with A-bimodule structure given by $a \cdot \xi \cdot b = \alpha(a)\xi b$. Define $J : H \to H : J(a) = \alpha(a)^*$. The n-fold tensor power $H^{\otimes_A^n}$ can be identified with $\mathbb{C}^2$ with the bimodule structure given by

$$a \cdot \xi \cdot b = \begin{cases} a\xi b & \text{if } n \text{ is even,} \\ \alpha(a)\xi b & \text{if } n \text{ is odd.} \end{cases}$$

We denote by $\{e_n, f_n\}$ the canonical orthonormal basis of $H^{\otimes_A^n}$ under this identification. For every $0 < \delta < 1$, denote by τ_δ the trace on A given by $\tau_\delta(a,b) = \delta a + (1-\delta)b$. With respect to the canonical trace $\tau = \tau_{1/2}$, the left and right creation operators associated with the identity $1 \in A = H$ then become

$$\ell(e_n) = e_{n+1} \quad , \quad \ell(f_n) = f_{n+1} \quad , \quad r(e_n) = f_{n+1} \quad , \quad r(f_n) = e_{n+1} \quad ,$$

for all $n \geq 0$.

By symmetry, it suffices to consider the case $0 < \delta \leq 1/2$. With respect to the trace τ_δ, the left and right creation operators ℓ_δ and r_δ can be realized on the same Hilbert space and are given by

$$\ell_\delta = \ell\, \lambda(D^{-1/2}) \quad \text{and} \quad r_\delta = r\, \rho(D^{-1/2}) \quad ,$$

where $D = (2\delta, 2(1-\delta))$ is the Radon-Nikodym derivative between τ_δ and $\tau_{1/2}$ and where we denote by $\lambda(\,\cdot\,)$ and $\rho(\,\cdot\,)$ the left, resp. right, action of A. Then,

$$M_\delta := \Gamma(H, J, A, \tau_\delta)'' = \lambda(A) \vee \{\ell_\delta + \ell_\delta^*\}'' = \lambda(A) \vee \{S_\delta\}'' \quad ,$$

where $S_\delta = \ell\lambda(\Delta^{-1/4}) + \ell^*\lambda(\Delta^{1/4})$ and $\Delta = (\delta/(1-\delta), (1-\delta)/\delta)$. We still denote by τ_δ the canonical trace on M_δ.

Note that $S_\delta = S_\delta^*$. Denoting by $e = (1,0)$ and $f = (0,1)$ the minimal projections in A, we have that $S_\delta e = f S_\delta$. When $\delta = 1/2$, the operator S_δ is nonsingular and diffuse. When $0 < \delta < 1/2$, the kernel of S_δ has dimension 1 and S_δ is diffuse on its orthogonal complement. We denote by z_δ the projection onto the kernel of S_δ. Then z_δ is a minimal and central projection in M_δ with $\tau_\delta(z_\delta) = 1 - 2\delta$. We conclude that there is a trace preserving $*$-isomorphism

$$(M_\delta, \tau_\delta) \cong \underbrace{M_2(\mathbb{C}) \otimes B_0}_{\delta(\mathrm{Tr} \otimes \tau_0)} \oplus \underbrace{\mathbb{C}}_{1-2\delta} \tag{3.4}$$

where (B_0, τ_0) is a diffuse abelian von Neumann algebra with normal faithful tracial state τ_0 and where we emphasized the choice of trace at the right hand side. Under the isomorphism (3.4), we have that

$$e \mapsto (e_{11} \otimes 1) \oplus 0 \quad , \quad f \mapsto (e_{22} \otimes 1) \oplus 1 \quad , \quad S_\delta \mapsto ((e_{12} + e_{21}) \otimes b) \oplus 0 \quad , \quad z_\delta \mapsto 0 \oplus 1$$

where $b \in B$ is a positive nonsingular element generating B.

Next, taking $H \oplus H$ and $J \oplus J$, it follows from Proposition 3.4 that

$$\mathcal{M}_\delta := \Gamma(H \oplus H, J \oplus J, A, \tau_\delta)'' = M_\delta *_A M_\delta \quad ,$$

where we used at the right hand side the amalgamated free product w.r.t. the unique τ_δ-preserving conditional expectations. We denote with superscripts $^{(1)}$ and $^{(2)}$ the elements of M_δ viewed in the first, resp. second copy of M_δ in the amalgamated free product. Note that $f^{(1)} = f^{(2)}$ and that, denoting this projection as f, we get that $fM_\delta^{(1)}f$ and $fM_\delta^{(2)}f$ are free inside $f\mathcal{M}_\delta f$. It then follows from [Vo86] that the projection $z := z_\delta^{(1)} \wedge z_\delta^{(2)}$ is nonzero if and only if $\delta < 1/3$. Using the diffuse subalgebras $B^{(1)}$ and $B^{(2)}$, we get that $\mathcal{Z}(\mathcal{M}_\delta) = \mathbb{C}z + \mathbb{C}(1-z)$. We conclude that $\Gamma(H \oplus H, J \oplus J, A, \tau_\delta)''$ is a factor if and only if $1/3 \leq \delta \leq 2/3$.

4 Normalizers and (relative) strong solidity

The main result of this section is the following dichotomy theorem for A-valued semicircular systems. In the special case of free Bogoljubov crossed products (see Remark 3.5), this result was proven in [Ho12b, Theorem B]. As explained in the introduction, the A-valued semicircular systems fit perfectly into Popa's deformation/rigidity theory. The proof of Theorem 4.1 therefore follows closely [IPP05, HS09, HR10, Io12, Ho12b], using in the same way Popa's s-malleable deformation given by "doubling and rotating" the initial A-bimodule ${}_A H_A$ (see below).

We freely use Popa's intertwining-by-bimodules (see [Po03, Section 2]) and the notion of relative amenability (see [OP07, Section 2.2]).

Theorem 4.1. *Let (A, τ) be a tracial von Neumann algebra and (H, J) a symmetric A-bimodule. Put $M = \Gamma(H, J, A, \tau)''$. Let $q \in M$ be a projection and $B \subset qMq$ a von Neumann subalgebra. If B is amenable relative to A, then at least one of the following statements holds: $B \prec_M A$ or $\mathcal{N}_M(B)''$ stays amenable relative to A.*

As a consequence of Theorem 4.1, we get the following strong solidity theorem.

Corollary 4.2. *Let (A, τ) be a tracial von Neumann algebra and (H, J) a symmetric A-bimodule. Denote $M = \Gamma(H, J, A, \tau)''$. Assume that ${}_A H_A$ is mixing.*

If $B \subset M$ is a diffuse von Neumann subalgebra that is amenable relative to A, then $\mathcal{N}_M(B)''$ stays amenable relative to A.

So if A is amenable and ${}_A H_A$ is mixing, we get that M is strongly solid.

Proof. Denote $P := \mathcal{N}_M(B)''$. Since $B \vee (B' \cap M) \subset P$, we have $P' \cap M = \mathcal{Z}(P)$. Denote by $z \in \mathcal{Z}(P)$ the smallest projection such that $Pz \not\prec_M A$. Then, $P(1 - z)$ fully embeds into A inside M and, in particular, $P(1 - z)$ is amenable relative to A. It remains to prove that also Pz is amenable relative to A.

Since the bimodule ${}_A H_A$ is mixing, the inclusion $A \subset M$ is mixing in the sense of [Po03, Proof of Theorem 3.1] and [Io12, Definition 9.2]. Since $\mathcal{N}_{zMz}(Bz)'' = Pz$, since Bz is diffuse and since $Pz \not\prec_M A$, it follows from [Io12, Lemma 9.4] that $Bz \not\prec_M A$. It then follows from Theorem 4.1 that Pz is amenable relative to A. $\square$

To prove Theorem 4.1, we fix a tracial von Neumann algebra (A, τ) and a symmetric A-bimodule (H, J). Put $M = \Gamma(H, J, A, \tau)''$ as in Definition 3.1. Recall that $L^2(M) = \mathcal{F}_A(H) = L^2(A) \oplus \bigoplus_{n=1}^{\infty} H^{\otimes_A^n}$.

We construct as follows an s-malleable deformation of M in the sense of [Po03]. Put

$$\mathcal{M} = \Gamma(H \oplus H, A, J \oplus J)'' .$$

By Proposition 3.4, we have $\mathcal{M} = M *_A M$. We denote by π_1 and π_2 the two canonical embeddings of M into $\mathcal{M}$. When no embedding is explicitly mentioned, we will always consider $M \subset \mathcal{M}$ via the embedding π_1.

Let $U_t \in \mathcal{U}(H \oplus H)$, $t \in \mathbb{R}$, be the rotation with angle t, i.e.,

$$U_t(\xi, \eta) = (\cos(t)\xi - \sin(t)\eta, \sin(t)\xi + \cos(t)\eta) \quad \text{for } \xi, \eta \in H .$$

Since the construction of $\Gamma(H, J, A, \tau)''$ is functorial, this gives rise to an automorphism $\theta_t := \operatorname{Ad} U_t \in \operatorname{Aut}(\mathcal{M})$. Note that $\theta_{\pi/2} \circ \pi_1 = \pi_2$.

Define $\beta \in \mathcal{U}(H)$ by $\beta(\xi, \eta) = (\xi, -\eta)$ for $\xi, \eta \in H$. Again by functoriality, we have that β defines an automorphism of $\mathcal{M}$. Now, β satisfies $\beta(x) = x$ for all $x \in \pi_1(M)$, $\beta^2 = \mathrm{id}$ and $\beta \circ \theta_t = \theta_{-t} \circ \beta$ for all t. Hence $(\mathcal{M}, (\theta_t)_{t \in \mathbb{R}})$ is an s-malleable deformation of M.

The following two lemmas are the key ingredients in the proof of Theorem 4.1.

Lemma 4.3. *Let $q \in M$ be a projection and $P \subset qMq$ a von Neumann subalgebra. If $\theta_t(P) \prec_{\mathcal{M}} \pi_i(M)$ for some $i \in \{1, 2\}$ and some $t \in (0, \frac{\pi}{2})$, then $P \prec_M A$.*

Lemma 4.4. *Let $q \in M$ be a projection and $P \subset qMq$ a von Neumann subalgebra. If $\theta_t(P)$ is amenable relative to A inside $\mathcal{M}$ for all $t \in (0, \frac{\pi}{2})$, then P is amenable relative to A inside M.*

Before proving Lemma 4.3 and Lemma 4.4, we first show how Theorem 4.1 follows from these two lemmas and we deduce a relative strong solidity theorem for A-valued semicircular systems.

Proof of Theorem 4.1. Put $P = \mathcal{N}_{qMq}(B)''$. We apply [Va13, Theorem A] to the subalgebra $\theta_t(B) \subset M *_A M$ for a fixed $t \in (0, \frac{\pi}{2})$. Note that $\theta_t(B)$ is normalized by $\theta_t(P)$. So, we get that one of the following holds:

1. $\theta_t(B) \prec_{\mathcal{M}} A$.
2. $\theta_t(P) \prec_{\mathcal{M}} \pi_i(M)$ for some $i \in \{1, 2\}$.
3. $\theta_t(P)$ is amenable relative to A inside $\mathcal{M}$.

If 1 or 2 holds, it follows by Lemma 4.3 that $B \prec_M A$. So, if we assume that $B \not\prec_M A$, we get that $\theta_t(P)$ is amenable relative to A inside $\mathcal{M}$ for all $t \in (0, \frac{\pi}{2})$. It then follows from Lemma 4.4 that $P = \mathcal{N}_{qMq}(B)''$ is amenable relative to A inside M. □

Proof of Lemma 4.3

We now turn to the proof of Lemma 4.3. We first give a sketch of the proof. For each $k \in \mathbb{N}$, we let $p_k \in B(L^2M)$ denote the projection onto $H^{\otimes_A^k}$. Given a von Neumann subalgebra $P \subset qMq$, we first show that if $\theta_t(P) \prec_{\mathcal{M}} \pi_i(M)$ for some $i \in \{1, 2\}$ and some $t \in (0, \frac{\pi}{2})$, then P has "bounded tensor length", in the sense that there exists $k \in \mathbb{N}$ and $\delta > 0$ such that $\|\sum_{i=0}^k p_i(a)\|_2 \geq \delta$ for all $a \in \mathcal{U}(P)$ (see Lemma 4.6). Next, we reason exactly as in the proof of [Po03, Theorem 4.1]. Since θ_t converges uniformly to id on the unit ball of $p_i(M)$ for any fixed $i \in \mathbb{N}$, we get a $t \in (0, \frac{\pi}{2})$ and a nonzero partial isometry $v \in \mathcal{M}$ such that $\theta_t(a)v = va$ for all $a \in \mathcal{U}(P)$. Using the automorphism β, we can even obtain $t = \pi/2$, i.e., $\pi_2(a)v = v\pi_1(a)$ for all $a \in \mathcal{U}(P)$. Using results of [IPP05] on amalgamated free product von Neumann algebras, this implies that $P \prec_M A$.

For simplicity, we put $M_i = \pi_i(M) \subset \mathcal{M}$ for $i \in \{1, 2\}$. Note that

$$L^2(M_1) = L^2(A) \oplus \bigoplus_{k=1}^{\infty} (H \oplus 0)^{\otimes_A^k}, \quad L^2(M_2) = L^2(A) \oplus \bigoplus_{k=1}^{\infty} (0 \oplus H)^{\otimes_A^k},$$

as subspaces of $L^2(\mathcal{M}) = \mathcal{F}_A(H \oplus H)$. Denote by $e_{M_i} \in B(L^2(\mathcal{M}))$ the projection onto $L^2(M_i)$.

Lemma 4.5. *If $\mu_n \in L^2(M_1)$ is a bounded sequence such that $\lim_{n\to\infty} \|p_k(\mu_n)\| = 0$ for all $k \geq 0$, then for all $i = 1, 2$, $0 < t < \frac{\pi}{2}$, integers $a, b, c, d \geq 0$ and vectors $\xi_i, \eta_i, \gamma_i, \rho_i \in \mathcal{H} \oplus \mathcal{H}$, we have*

$$\lim_{n\to\infty} \|e_{M_i}(\ell(\xi_1) \cdots \ell(\xi_a)\ell(\eta_b)^* \cdots \ell(\eta_1)^* r(\gamma_c) \cdots r(\gamma_1) r(\rho_1)^* \cdots r(\rho_d)^* U_t \mu_n)\| = 0 .$$

Proof. Fix $t \in (0, \frac{\pi}{2})$ and define $\delta_1 = \cos t$ and $\delta_2 = \sin t$. Define the operator $Z_i \in B(L^2\mathcal{M})$ for $i = 1, 2$ by

$$Z_i = \bigoplus_{e \geq b+d} \delta_i^{e-b-d} (U_t^{\otimes_A^b} \otimes_A 1^{\otimes_A^{(e-b-d)}} \otimes_A U_t^{\otimes_A^d}) .$$

Denote $p_{\geq \kappa} = \sum_{i=\kappa}^{\infty} p_i$ and $p_{<\kappa} = \sum_{i=0}^{k-1} p_i$. When $\kappa \geq b+d$, we have $\|Z_i p_{\geq\kappa}\| = \delta_i^{\kappa-b-d}$. Since $\lim_n \|p_{<\kappa}(\mu_n)\| = 0$ for every κ, we get that $\lim_n \|Z_i(\mu_n)\| = 0$. So, it suffices to prove that

$$\begin{aligned} &e_{M_i}(\ell(\xi_1)\cdots\ell(\xi_a)\ell(\eta_b)^*\cdots\ell(\eta_1)^* r(\gamma_c)\cdots r(\gamma_1) r(\rho_1)^*\cdots r(\rho_d)^* U_t p_{\geq b+d}(\mu)) \\ &\quad = \ell(q_i\xi_1)\cdots\ell(q_i\xi_a)\ell(\eta_b)^*\cdots\ell(\eta_1)^* r(q_i\gamma_c)\cdots r(q_i\gamma_1) r(\rho_1)^*\cdots r(\rho_d)^* Z_i(\mu) \end{aligned}$$

for all $\mu \in L^2(M_1)$, where q_1, resp. q_2, denotes the orthogonal projection of $H \oplus H$ onto $H \oplus 0$, resp. $0 \oplus H$. It is sufficient to check this formula for $\mu = \mu_1 \otimes_A \cdots \otimes_A \mu_e$ with $\mu_i \in \mathcal{H} \oplus 0$ and $e \geq b+d$, where it follows by a direct computation. □

Lemma 4.6. *If $a_n \in M$ is a bounded sequence with $\lim_n \|p_k(a_n)\|_2 = 0$ for all $k \geq 0$, then*

$$\lim_{n\to\infty} \|E_{M_i}(x\theta_t(a_n)y)\|_2 = 0\ ,$$

for all $i \in \{1,2\}$, $0 < t < \frac{\pi}{2}$ and $x, y \in \mathcal{M}$.

Proof. It suffices to take $x = W(\xi_1,\ldots,\xi_k)$ and $y = W(\eta_1,\ldots,\eta_m)$ with $\xi_i, \eta_i \in \mathcal{H} \oplus \mathcal{H}$ (as defined in Section 3), since these elements span a $\|\cdot\|_2$-dense subspace of $\mathcal{M} \ominus A$. Then,

$$\begin{aligned} &E_{M_i}(x\theta_t(a_n)y) = e_{M_i}(xJy^*JU_t(a_n\Omega)) \\ &= \sum_{s=0}^{k}\sum_{r=0}^{m} e_{M_i}(\ell(\xi_1)\cdots\ell(\xi_s)\ell(J\xi_{s+1})^*\cdots\ell(J\xi_k)^* r(\eta_m)\cdots r(\eta_{r+1}) r(J\eta_r)^*\cdots r(J\eta_1)^* U_t(a_n\Omega))\ , \end{aligned}$$

and the result now follows from Lemma 4.5 □

We are now ready to finish the proof of Lemma 4.3.

Proof of Lemma 4.3. Assume that $\theta_t(P) \prec M_i$ for some $i \in \{1,2\}$ and $t \in (0,\frac{\pi}{2})$. By Lemma 4.6, we get a $\delta > 0$ and $\kappa > 0$ such that $\|\sum_{i=0}^{\kappa} p_i(a)\|_2^2 \geq 2\delta$ for all $a \in \mathcal{U}(P)$. Note that $\langle U_t(p_i(a)), p_j(a)\rangle = 0$ if $i \neq j$ and that $\langle U_t(p_i(a)), p_i(a)\rangle = \cos(t)^i \|p_i(a)\|_2^2$. Choose $t_0 \in (0,\frac{\pi}{2})$ such that $\cos(t_0)^i \geq 1/2$ for all $i = 0,\ldots,\kappa$. Note that we may choose t_0 of the form $t_0 = \pi/2^n$. For all $a \in \mathcal{U}(P)$, we then have

$$\begin{aligned} \tau(\theta_{t_0}(a)a^*) = \langle U_{t_0}(a), a\rangle &= \sum_{i,j=0}^{\infty} \langle U_{t_0}(p_i(a)), p_j(a)\rangle = \sum_{i=0}^{\infty} \cos(t_0)^i \|p_i(a)\|_2^2 \\ &\geq \sum_{i=0}^{\kappa} \cos(t_0)^i \|p_i(a)\|_2^2 \geq \frac{1}{2} 2\delta = \delta\ . \end{aligned}$$

Let v be the unique element of minimal $\|\cdot\|_2$-norm in the $\|\cdot\|_2$-closed convex hull of $\{\theta_{t_0}(a)a^* \mid a \in \mathcal{U}(P)\}$. Then $v \in \mathcal{M}$ and $\theta_{t_0}(a)v = va$ for all $a \in \mathcal{U}(P)$. Moreover, $v \neq 0$ since $\tau(v) \geq \delta$.

Put $w_1 = \theta_{t_0}(v\beta(v^*))$. Then w_1 satisfies $w_1 a = \theta_{2t_0}(a)w_1$ for all $a \in \mathcal{U}(P)$. However, we do not know yet that w_1 is nonzero. Assuming that $P \not\prec_M A$, we have from Proposition 3.4 and [IPP05, Theorem 1.2.1] that $P' \cap q\mathcal{M}q \subset qMq$, hence $v^*v \in qMq$. Thus

$$w_1^* w_1 = \theta_{t_0}(\beta(v)v^*v\beta(v^*)) = \theta_{t_0}(\beta(vv^*)) \neq 0\ .$$

By iterating this process, we obtain $w = w_{n-1} \neq 0$ such that $wa = \theta_{\pi/2}(a)w$, i.e., $w\pi_1(a) = \pi_2(a)w$ for all $a \in P$. This means that $P \prec_{\mathcal{M}} M_2$. As in [Ho07, Claim 5.3]), this is incompatible with our assumption $P \not\prec_M A$. So it follows that $P \prec_M A$ and the lemma is proved. □

Proof of Lemma 4.4

Proof. Let $P \subset qMq$ and assume that $\theta_t(P)$ is amenable relative to A in $\mathcal{M}$ for all $t \in (0, \frac{\pi}{2})$. As in the proof of [Io12, Theorem 5.1] (and [Va13, Theorem 3.4]), we let I be the set of all quadruples $i = (X, Y, \delta, t)$ where $X \subset \mathcal{M}$ and $Y \subset \mathcal{U}(P)$ are finite subsets, $\delta \in (0,1)$ and $t \in (0, \frac{\pi}{2})$. Then I is a directed set when equipped with the ordering $(X, Y, \delta, t) \leq (X', Y', \delta', t')$ if and only if $X \subset X'$, $Y \subset Y'$, $\delta' \leq \delta$ and $t' \leq t$.

By [OP07, Theorem 2.1], we can for each $i = (X, Y, \delta, t) \in I$ choose a vector $\xi_i \in \theta_t(q) L^2(\mathcal{M}) \otimes_A L^2(\mathcal{M}) \theta_t(q)$ such that $\|\xi_i\|_2 \leq 1$ and

$$\begin{aligned} &|\langle x\xi_i, \xi_i\rangle - \tau(x\theta_t(q))| \leq \delta \quad \text{for every } x \in X \text{ or } x = (\theta_t(y) - y)^*(\theta_t(y) - y) \text{ with } y \in Y \ , \\ &\|\theta_t(y)\xi_i - \xi_i\theta_t(y)\|_2 \leq \delta \quad \text{for every } y \in Y \ . \end{aligned}$$

We now prove that ${}_{qMq}L^2(qMq)_P$ is weakly contained in ${}_{qMq}(qL^2(\mathcal{M}) \otimes_A L^2(\mathcal{M})q)_P$. For this, it suffices to show that

$$\begin{aligned} &\lim_i \langle x\xi_i, \xi_i\rangle = \tau(x) \quad \text{for every } x \in q\mathcal{M}q \ , \\ &\lim_i \|y\xi_i - \xi_i y\|_2 = 0 \quad \text{for every } y \in P \ . \end{aligned} \tag{4.1}$$

Let $y \in \mathcal{U}(P)$ and $\varepsilon > 0$ be given. Choose $t > 0$ small enough so that $\|\theta_t(y) - y\|_2^2 \leq \varepsilon/6$. We have

$$\|y\xi_i - \xi_i y\|_2 \leq \|(y - \theta_t(y))\xi_i\|_2 + \|\theta_t(y)\xi_i - \xi_i\theta_t(y)\|_2 + \|\xi_i(\theta_t(y) - y)\|_2$$

for any $i \in I$. Moreover,

$$\|(y - \theta_t(y))\xi_i\|_2^2 = \langle(\theta_t(y) - y)^*(\theta_t(y) - y)\xi_i, \xi_i\rangle \leq \|(\theta_t(y) - y)\theta_t(q)\|_2^2 + \varepsilon/6 \leq \varepsilon/3 \ ,$$

for $i \geq (\{0\}, \{y\}, \varepsilon/6, t)$ in I. Similarly, we get that $\|\xi_i(\theta_t(y) - y)\|_2 \leq \varepsilon/3$. Thus, we conclude that $\|y\xi_i - \xi_i y\|_2 \leq \varepsilon$ for $i \geq (\{0\}, \{y\}, \varepsilon/6, t)$ and so the second assertion of (4.1) holds true. The first assertion is proved similarly, using that $\|\theta_t(q) - q\|_2 \to 0$ as $t \to 0$.

By Proposition 3.4, we have $\mathcal{M} = M_1 *_A M_2$. Under our identification $M = M_1$, we then get that ${}_M L^2(\mathcal{M})_A \cong {}_M(L^2(M) \otimes_A \mathcal{K})_A$, where ${}_A\mathcal{K}_A$ is the A-bimodule defined as the direct sum of $L^2(A)$ and all alternating tensor products $L^2(M_2 \ominus A) \otimes_A L^2(M_1 \ominus A) \otimes_A \cdots$ starting with $L^2(M_2 \ominus A)$. We conclude that ${}_{qMq}L^2(qMq)_P$ is weakly contained in ${}_{qMq}(qL^2(M) \otimes_A (\mathcal{K} \otimes_A L^2(\mathcal{M})q))_P$. It then follows from [PV11, Proposition 2.4] that P is amenable relative to A inside M. This finishes the proof of Lemma 4.4. $\square$

5 Maximal amenability

Fix a tracial von Neumann algebra (A, τ) and a symmetric Hilbert A-bimodule ${}_AH_A$ with symmetry $J : H \to H$. Denote by $M = \Gamma(H, J, A, \tau)''$ the associated von Neumann algebra with faithful normal tracial state τ. We prove the following maximal amenability property by combining Popa's asymptotic orthogonality [Po83] with the method of [BH16]. In the special case of free Bogoljubov crossed products (see Remark 3.5), part 3 of Theorem 5.1 was proved in [Ho12b, Theorem D].

Theorem 5.1. *Assume that ${}_AH_A$ is weakly mixing. Then the following properties hold.*

1. $\mathcal{Z}(M) = \{a \in \mathcal{Z}(A) \mid a\xi = \xi a \text{ for all } \xi \in H\}$.

2. *If $B \subset M$ is a von Neumann subalgebra that is amenable relative to A inside M and if the bimodule ${}_{B \cap A}H_A$ is left weakly mixing, then $B \subset A$.*

3. *A von Neumann subalgebra of M that properly contains A is not amenable relative to A inside M. If the A-bimodule ${}_AH_A$ is faithful[2], then M has no amenable direct summand. If A is amenable, then $A \subset M$ is a maximal amenable subalgebra.*

Proof. As above, identify

$$L^2(M) = L^2(A) \oplus \bigoplus_{n \geq 1} (\underbrace{H \otimes_A \cdots \otimes_A H}_{n\text{-fold}})$$

and denote by $\mathcal{H} \subset H$ the subspace of vectors that are both left and right bounded.

1. Since ${}_AH_A$ is weakly mixing, it follows from Proposition 2.3 that the n-fold tensor products $H \otimes_A \cdots \otimes_A H$ (with $n \geq 1$) have no A-central vectors. Therefore, $A' \cap M = \mathcal{Z}(A)$. Looking at the commutator of $a \in \mathcal{Z}(A)$ and $\ell(\xi) + \ell(J\xi)^*$, the conclusion follows.

2. Since B is amenable relative to A inside M, we can fix a B-central state $\omega \in \langle M, e_A \rangle^*$ such that $\omega|_M = \tau$.

Claim I. For every $\xi \in \mathcal{H}$ and every $\varepsilon > 0$, there exists a projection $p \in A$ such that $\tau(1-p) < \varepsilon$ and such that

$$\omega(\ell(\xi p)\ell(\xi p)^*) < \varepsilon \; .$$

To prove this claim, fix $\xi \in \mathcal{H}$ and $\varepsilon > 0$. Define $a = \sqrt{\langle \xi, \xi \rangle_A}$ and denote by $q \in A$ the support projection of a. Take a projection $q_1 \in qAq$ that commutes with a, such that $\tau(q - q_1) < \varepsilon/2$ and such that aq_1 is invertible in q_1Aq_1. Denote by $b \in q_1Aq_1$ this inverse and define $\eta = \xi b$. By construction, $\ell(\eta)^*\ell(\eta) = q_1$ and $\xi q_1 = \eta a$.

Pick a positive integer N such that $2^{-N} < \varepsilon/(2\|a\|^2)$. Put $\kappa = 2^N$. Then pick $\delta > 0$ such that $\delta < \varepsilon/(\kappa 2\|a\|^2)$. We start by constructing unitary operators $v_1, \ldots, v_\kappa \in \mathcal{U}(A \cap B)$ and a projection $q_2 \in q_1Aq_1$ such that $\tau(q_1 - q_2) < \varepsilon/2$ and such that the vectors $\eta_i = v_i\eta$ satisfy

$$\|q_2 \langle \eta_i, \eta_j \rangle_A \, q_2\| < \delta \quad \text{whenever } \; i \neq j \tag{5.1}$$

(and where we indeed use the operator norm at the left hand side of (5.1)).

We put $e_0 = q_1$ and $v_1 = 1$. Denoting by (a_i) the net of unitaries in $B \cap A$ witnessing the left weak mixing of ${}_{B \cap A}H_A$, we get that $\lim_i \|\langle \eta, a_i\eta \rangle_A\|_2 = 0$. So we find a net of projections $r_i \in q_1Aq_1$ such that $\tau(q_1 - r_i) \to 0$ and

$$\|r_i \, \langle \eta, a_i\eta \rangle_A \, r_i\| < \delta \quad \text{for every } i.$$

Take i large enough such that $\tau(q_1 - r_i) < \varepsilon/4$ and define $e_1 := r_i$ and $v_2 := a_i$. We have now constructed v_1, v_2. Inductively, we double the length of the sequence, until we arrive at $v_1, \ldots, v_\kappa$. After k steps, we have constructed the projections $e_1 \geq \cdots \geq e_k$ and unitaries $v_1, \ldots, v_{2^k}$ in $\mathcal{U}(B \cap A)$ such that $\tau(e_{j-1} - e_j) < 2^{-j-1}\varepsilon$ and such that the vectors $\eta_i = v_i\eta$ satisfy

$$\|e_k \, \langle \eta_i, \eta_j \rangle_A \, e_k\| < \delta \quad \text{whenever } \; i \neq j \; .$$

As in the first step, we can pick a unitary $a \in \mathcal{U}(B \cap A)$ and a projection $e_{k+1} \in e_kAe_k$ such that $\tau(e_k - e_{k+1}) < 2^{-k-2}\varepsilon$ and such that

$$\|e_{k+1} \, \langle \eta_i, a\eta_j \rangle_A \, e_{k+1}\| < \delta$$

[2] A P-Q-bimodule ${}_PH_Q$ is called faithful if the $*$-homomorphisms $P \to B(H)$ and $Q^{\mathrm{op}} \to B(H)$ are faithful.

for all $i, j \in \{1, \dots, 2^k\}$. It now suffices to put $v_{2^k+i} = a v_i$ for all $i = 1, \dots, 2^k$. We have doubled our sequence. We continue for N steps and put $q_2 = e_N$. So, (5.1) is proved.

Put $\mu_i = \eta_i q_2 = v_i \eta q_2$. Define the projections $P_i = \ell(\mu_i)\ell(\mu_i)^*$ and note that $P_i = v_i P_1 v_i^*$. By construction, $\|P_i P_j\| < \delta$ whenever $i \neq j$. Writing $P = \sum_{i=1}^{\kappa} P_i$ it follows that $\|P^2 - P\| < \kappa^2 \delta$. Since P is a positive operator, we conclude that $\|P\| < 1 + \kappa^2\delta$. Since ω is B-central and $v_i \in B$ for all i, we get that

$$\kappa\, \omega(P_1) = \sum_{i=1}^{\kappa} \omega(P_i) = \omega(P) \leq \|P\| < 1 + \kappa^2 \delta \; .$$

Therefore, $\omega(P_1) < \kappa^{-1} + \kappa\delta < \|a\|^{-2}\varepsilon$.

Since q_1 and a commute, the right support of $(q_1 - q_2)a$ is a projection of the form $q_1 - p_0$ where $p_0 \in q_1 A q_1$ is a projection with $\tau(q_1 - p_0) \leq \tau(q_1 - q_2) < \varepsilon/2$. By construction, $q_1 a p_0 = q_2 a p_0$. Since $p_0 \leq q_1$ and $\eta = \eta q_1$, it follows that

$$\xi p_0 = \xi q_1 p_0 = \eta a p_0 = \eta q_1 a p_0 = \eta q_2 a p_0 \; .$$

Define the projection $p \in A$ given by $p = (1 - q) + p_0$. Since $\xi(1 - q) = 0$, we still have $\xi p = \eta q_2 a p_0$. Because $1 - p = (q - q_1) + (q_1 - p_0)$, we get that $\tau(1 - p) < \varepsilon$. Finally,

$$\omega(\ell(\xi p)\ell(\xi p)^*) = \omega(\ell(\eta q_2)\, a p_0 a^*\, \ell(\eta q_2)^*) \leq \|a\|^2\, \omega(\ell(\eta q_2)\ell(\eta q_2)^*) = \|a\|^2\, \omega(P_1) < \varepsilon \; .$$

So, we have proven Claim I.

Claim II. For every $\xi \in \mathcal{H}$ and every $\varepsilon > 0$, there exists a projection $p \in A$ such that $\tau(1 - p) < \varepsilon$ and such that $\omega(\ell(\xi p)\ell(\xi p)^*) = 0$.

For every integer $k \geq 1$, Claim I gives a projection $p_k \in A$ with $\tau(1 - p_k) < 2^{-k}\varepsilon$ and $\omega(\ell(\xi p_k)\ell(\xi p_k)^*) < 1/k$. Defining $p = \bigwedge_k p_k$, we get that $\tau(1 - p) < \varepsilon$ and, for every $k \geq 1$,

$$\omega(\ell(\xi p)\ell(\xi p)^*) = \omega(\ell(\xi) p \ell(\xi)^*) \leq \omega(\ell(\xi) p_k \ell(\xi)^*) = \omega(\ell(\xi p_k)\ell(\xi p_k)^*) < 1/k \; .$$

So, $\omega(\ell(\xi p)\ell(\xi p)^*) = 0$ and claim II is proved.

We can now conclude the proof of 2. Denote by $E_A : M \to A$ and $E_B : M \to B$ the unique trace preserving conditional expectations. It is sufficient to prove that $E_B \circ E_A = E_B$. So we have to prove that $E_B(x) = 0$ for all $x \in M \ominus A$. Using the Wick products defined in (3.2), it suffices to prove that $E_B(W(\xi_1, \dots, \xi_k)) = 0$ for all $k \geq 1$ and all $\xi_1, \dots, \xi_k \in \mathcal{H}$.

Since ω is B-central and $\omega|_M = \tau$, there is a unique conditional expectation $\Phi : \langle M, e_A \rangle \to B$ such that $\Phi|_M = E_B$ and $\omega = \tau \circ \Phi$.

We first consider $k \geq 2$ and $\xi_1, \dots, \xi_k \in \mathcal{H}$. By Claim II, we can take sequences of projections $p_n, q_n \in A$ such that $p_n \to 1$ and $q_n \to 1$ strongly and

$$\Phi(\ell(\xi_1 p_n)\ell(\xi_1 p_n)^*) = 0 = \Phi(\ell((J\xi_k)q_n)\ell((J\xi_k)q_n)^*)$$

for all n. Then also $\Phi(\ell(\xi_1 p_n)T) = 0 = \Phi(T\ell((J\xi_k)q_n)^*)$ for all n and all $T \in \langle M, e_A \rangle$. We conclude that

$$E_B(W(\xi_1 p_n, \xi_2, \dots, \xi_{k-1}, q_n \xi_k)) = \Phi(W(\xi_1 p_n, \xi_2, \dots, \xi_{k-1}, q_n \xi_k)) = 0$$

for all n. Since E_B is normal, it follows that $E_B(W(\xi_1, \dots, \xi_k)) = 0$.

We next consider the case $k = 1$. So it remains to prove that $E_B(\ell(\xi) + \ell(J\xi)^*) = 0$ for all $\xi \in \mathcal{H}$. For this, it is sufficient to prove that $\Phi(\ell(\xi)) = 0$ for all $\xi \in \mathcal{H}$. By Claim II and reasoning as above, we find a sequence of projections $p_n \in A$ such that $p_n \to 1$ strongly and

$\Phi(\ell(\xi p_n)T) = 0$ for all n and all $T \in \langle M, e_A\rangle$. In particular, we can take $T = 1$ and get that $\Phi(\ell(\xi)p_n) = 0$ for all n. Write $e_n = 1 - p_n$. Then,

$$\Phi(\ell(\xi))^*\Phi(\ell(\xi)) = \Phi(\ell(\xi)e_n)^*\Phi(\ell(\xi)e_n) \leq \|\ell(\xi)\|^2 \,\Phi(e_n) = \|\ell(\xi)\|^2 \,E_B(e_n) \;.$$

Since $E_B(e_n) \to 0$ strongly, we conclude that $\Phi(\ell(\xi)) = 0$. This concludes the proof of 2.

3. It follows from 2 that a von Neumann subalgebra of M properly containing A is not amenable relative to A and thus, not amenable itself. Whenever $H \neq \{0\}$, we have $A \neq M$ and we conclude that M is not amenable. By 1, any direct summand of M is given as the von Neumann algebra associated with the symmetric weakly mixing Az-bimodule Hz where $z \in \mathcal{Z}(A)$ is a nonzero central projection satisfying $\xi z = z\xi$ for all $\xi \in H$. If ${}_A H_A$ is faithful, we have $Hz \neq \{0\}$ and it follows that this direct summand is not amenable. The final statement is an immediate consequence of 2. □

6 Absence of Cartan subalgebras

In this section, we give a complete description of the structure of the von Neumann algebra $M = \Gamma(H, J, A, \tau)''$ associated with an arbitrary symmetric A-bimodule (H, J). We describe the trivial direct summands of M and then prove that the remaining direct summand never has a Cartan subalgebra and describe its center (see Theorem 6.1). In all interesting cases, there are no trivial direct summands and this allows us to prove absence of Cartan subalgebras whenever H is a weakly mixing A-bimodule (Corollary 6.2), when A is a II_1 factor and H is not the trivial bimodule nor the bimodule given by a period 2 automorphism of A (Corollary 6.3), and finally for arbitrary free Bogoljubov crossed products (Corollary 6.4). This last result improves [Ho12b, Corollary C].

To prove our general structure theorem, we need the following terminology. Fix a tracial von Neumann algebra (A, τ). We say that an A-bimodule H is given by a partial automorphism if one of the following two equivalent conditions holds.

- The commutant of the right A action on H equals the left A action, and vice versa.
- There exists a projection $e \in B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} A$, a central projection $z \in \mathcal{Z}(A)$ and a surjective $*$-isomorphism $\alpha : Az \to e(B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} A)e$ such that ${}_A H_A \cong e(\ell^2(\mathbb{N}) \otimes L^2(A))$ with the bimodule structure given by $a \cdot \xi \cdot b = \alpha(a)\xi b$.

Fix a symmetric A-bimodule (H, J) and denote $M = \Gamma(H, J, A, \tau)''$. Then, M has two trivial direct summands. First denote by $z_0 \in \mathcal{Z}(A)$ the largest projection such that $z_0 H = \{0\}$. Then, $z_0 \in \mathcal{Z}(M)$ and $M z_0 = A z_0$. Next, there is a largest projection $z_1 \in \mathcal{Z}(A)(1 - z_0)$ such that $z_1 H = H z_1$ and such that the A-bimodule $H z_1$ is given by a partial automorphism of A (see Lemma 6.6 for details). Again $z_1 \in \mathcal{Z}(M)$ and $M z_1$ can be computed by the methods of Example 3.6. In a way, $M z_1$ is not very interesting, since it is always a direct sum of a corner of A and a corner of $A \mathbin{\overline{\otimes}} L^\infty([0, 1])$ or of an index 2 extension of this.

Writing $z_2 = 1 - (z_0 + z_1)$, we thus get that

$$M = A z_0 \oplus \Gamma(H z_1, J, A z_1, \tau)'' \oplus \Gamma(H z_2, J, A z_2, \tau)''$$

and only the third direct summand is “interesting and nontrivial”. By Lemma 6.6, the symmetric $A z_2$-bimodule $H z_2$ is *completely nontrivial* in the following sense: the left action of $A z_2$ on H is faithful and there are no nonzero projections $e, f \in \mathcal{Z}(A) z_2$ such that $eH = Hf$ and such that eH is given by a partial automorphism of $A z_2$. So it suffices to describe the structure of the von Neumann algebra associated with an arbitrary completely nontrivial symmetric A-bimodule.

We denote by $\dim_{-A}(K)$ the right A-dimension of a right Hilbert A-module K. Recall that the value of $\dim_{-A}(K)$ depends on the choice of the trace τ. We similarly define $\dim_{A-}(K)$ for a left Hilbert A-module K. As in (6.10), for every A-bimodule H, there is a unique element Δ_H^ℓ in the extended positive part of $\mathcal{Z}(A)$ characterized by $\tau(\Delta_H^\ell e) = \dim_{-A}(eH)$ for every projection $e \in \mathcal{Z}(A)$.

Theorem 6.1. *Let (A, τ) be a tracial von Neumann algebra and (H, J) a completely nontrivial symmetric A-bimodule. Write $M = \Gamma(H, J, A, \tau)''$. There is a canonical central projection $q \in \mathcal{Z}(M)$ (which, most of the time, is zero) such that the following holds.*

(a) No direct summand of $M(1-q)$ is amenable relative to $A(1-q)$.

(b) No direct summand of $M(1-q)$ admits a Cartan subalgebra.

(c) $Mq = Aq$ and the support of $E_A(1-q)$ equals 1.

(d) Defining $C := \{a \in \mathcal{Z}(A) \mid a\xi = \xi a \text{ for all } \xi \in H\}$, we get that $\mathcal{Z}(M) = \mathcal{Z}(A)q + C(1-q)$.

Moreover, we have that $E_A(q) = Z(\Delta_H^\ell)$, where $Z : (0, +\infty) \to \mathbb{R}$ is the positive function given by $Z(t) = 1 - t$ when $t \in (0, 1)$ and $Z(t) = 0$ when $t \geq 1$.

Corollary 6.2. *Let (A, τ) be a tracial von Neumann algebra and (H, J) a symmetric A-bimodule. Put $M = \Gamma(H, J, A, \tau)''$. If ${}_A H_A$ is weakly mixing and faithful, then no direct summand of M has a Cartan subalgebra and $\mathcal{Z}(M) = \{a \in \mathcal{Z}(A) \mid a\xi = \xi a \text{ for all } \xi \in H\}$.*

Proof. Let $z \in \mathcal{Z}(A)$ be a nonzero central projection. Since $zH \neq \{0\}$ and zH is still left weakly mixing as an A-bimodule, we have that $\dim_{-A}(zH) = +\infty$ and that zH is not given by a partial automorphism of A. So the conclusions follow from Theorem 6.1. □

When A is a II_1 factor, the results of Theorem 6.1 can be formulated more easily as follows.

Corollary 6.3. *Let A be a II_1 factor with its unique tracial state τ and let (H, J) be a symmetric A-bimodule. Denote $M = \Gamma(A, \tau, H, J)''$. Unless H is zero or H is the trivial A-bimodule or H is the symmetric A-bimodule associated with a period 2 outer automorphism of A, the following holds: M is a factor, M is not amenable relative to A and M has no Cartan subalgebra.*

Proof. Since A is a II_1 factor, the only symmetric A-bimodules given by a partial automorphism of A are the trivial A-bimodule and the A-bimodule given by $\alpha \in \mathrm{Aut}(A)$ with $\alpha \circ \alpha$ being inner. When a symmetric A-bimodule H is not given by a partial automorphism of A, we have that $\dim_{-A}(H) > 1$. So, the conclusion follows from Theorem 6.1. □

We finally deduce that free Bogoljubov crossed products never have a Cartan subalgebra. In [Ho12b, Corollary C], this was proven under extra assumptions on the underlying orthogonal representation.

Corollary 6.4. *Let G be an arbitrary countable group and $\pi : G \to O(K_\mathbb{R})$ an orthogonal representation of G with $\dim(K_\mathbb{R}) \geq 2$. Denote by $\sigma_\pi : G \curvearrowright \Gamma(K_\mathbb{R})'' \cong L(\mathbb{F}_{\dim K_\mathbb{R}})$ the associated free Bogoljubov action with crossed product $M := \Gamma(K_\mathbb{R})'' \rtimes^{\sigma_\pi} G$ (see Remark 3.5). Then no direct summand of M has a Cartan subalgebra. Also, M is a factor if and only if $\pi(g) \neq 1$ for every $g \in G \setminus \{e\}$ that has a finite conjugacy class.*

Proof. Write $A = L(G)$ with its canonical tracial state τ. By Remark 3.5, we can view $M = \Gamma(H, J, A, \tau)''$ where the symmetric A-bimodule (H, J) is given by (3.3). Denote by K the complexification of $K_\mathbb{R}$. Observe that $H \cong \ell^2(G) \otimes K$ with bimodule structure $a \cdot \xi \cdot b =$

$\alpha(a)\xi b$, where $\alpha : L(G) \to L(G) \mathbin{\overline{\otimes}} B(K)$ is given by $\alpha(u_g) = u_g \otimes \pi(g)$ for all $g \in G$. Since $(\tau \otimes \mathrm{id})\alpha(a) = \tau(a)1$ for all $a \in L(G)$, it follows that $\Delta_H^\ell = \dim(K_\mathbb{R})\, 1$.

The left and right actions of A on H are faithful. Since $H \otimes_A \overline{H}$ can be identified with the bimodule associated with the representation $\pi \otimes \overline{\pi}$, the center valued dimension of $H \otimes_A \overline{H}$ as a left A-module equals $\dim(K_\mathbb{R})^2\, 1$. It follows from Lemma 6.5 below that H is completely nontrivial. So, all conclusions follow from Theorem 6.1. □

We now prove Theorem 6.1, using several lemmas that we prove at the end of this section.

Proof of Theorem 6.1. Let $K \subset H$ be the maximal left weakly mixing A-subbimodule of H, i.e. the orthogonal complement of the span of all A-subbimodules of H having finite right A-dimension. Denote by $z_0 \in \mathcal{Z}(A)$ the support of the left A action on K. In the first part of the proof, assuming $z_0 \neq 0$, we show that

(1) $\mathcal{Z}(M) z_0 \subset \mathcal{Z}(A) z_0$,

(2) every M-central state ω on $\langle M, e_A\rangle$ that is normal on M satisfies $\omega(z_0) = 0$.

Note that $K \subset z_0 H$. Denote by $\mathcal{K} \subset K$ the dense subspace of vectors that are both left and right bounded. Define the von Neumann subalgebra $N \subset z_0 M z_0$ given by

$$N := \bigl(A z_0 \cup \{W(\xi, J(\mu)) \mid \xi, \mu \in \mathcal{K}\}\bigr)'' , \tag{6.1}$$

where we used the notation of (3.2). Then, the linear span of Az_0 and elements of the form $W(\xi_1, J(\mu_1), \ldots, \xi_k, J(\mu_k))$, $k \geq 1$, $\xi_i, \mu_i \in \mathcal{K}$, is a dense $*$-subalgebra of N.

Whenever $K_1, \ldots, K_n \subset H$ are A-subbimodules, we denote by concatenation $K_1 \cdots K_n$ the A-subbimodule of $L^2(M)$ given by

$$K_1 \cdots K_n := K_1 \otimes_A \cdots \otimes_A K_n \subset H \otimes_A \cdots \otimes_A H \subset L^2(M) .$$

In the same way, we write powers of A-subbimodules and when $K_i \subset H^{k_i}$ are A-subbimodules, then $K_1 \cdots K_n \subset H^{k_1 + \cdots + k_n}$ is a well defined A-subbimodule.

Using this notation, note that $L^2(N)$ is the direct sum of $L^2(Az_0)$ and the spaces $L_n := (K\, J(K))^n$, $n \geq 1$. Since K is a left weakly mixing A-bimodule, it follows that $N \cap (Az_0)' = \mathcal{Z}(A) z_0$.

We claim that

(3) $N \not\prec_N Az_0$, meaning that the N-A-bimodule $L^2(N)$ is left weakly mixing.

Since $N \cap (Az_0)' = \mathcal{Z}(A) z_0$, to prove this claim, it suffices to show that $\dim_{-A}(L^2(N)e) = +\infty$ for every nonzero projection $e \in \mathcal{Z}(A) z_0$. Since the left action of Az_0 on K is faithful and K is left weakly mixing, we get that $\dim_{-A}(K\, J(K)\, e) = +\infty$. So certainly $\dim_{-A}(L^2(N)e) = +\infty$ and the claim follows.

Proof of (1). Define the A-subbimodule $R \subset L^2(M)$ given as

$$R := \bigl(H \ominus \overline{(K + J(K))} \bigr) \oplus \bigoplus_{n=0}^{\infty} (H \ominus K)\, H^n\, (H \ominus J(K)) .$$

Since K is left weakly mixing and $J(K)$ is right weakly mixing, all A-central vectors in $L^2(M)$ belong to $L^2(A) + R$. Next note that left, resp. right multiplication by elements of N induces an N-bimodular unitary operator

$$L^2(N) \otimes_A R \otimes_A L^2(N) \to \overline{NRN} \subset L^2(z_0 M z_0) .$$

Since the N-A-bimodule $L^2(N)$ is left weakly mixing, it follows that $\overline{NRN}$ has no nonzero N-central vectors. Every element $x \in \mathcal{Z}(M)z_0$ defines a vector in $L^2(z_0 M z_0)$ that is both A-central and N-central. By A-centrality, we conclude that $x \in Az_0 + z_0 R z_0$. In particular, $x \in L^2(N) + \overline{NRN}$. Since x is N-central and $\overline{NRN}$ has no nonzero N-central vectors, we get that $x \in L^2(N)$ and thus, $x \in \mathcal{Z}(A)z_0$.

Proof of (2). Denote $L_{\text{even}} := L^2(N)$ and define L_{odd} as the direct sum of the A-bimodules $(K\, J(K))^n\, K$, $n \geq 0$. Note that both L_{even} and L_{odd} are N-A-bimodules. The same argument as in the proof of Theorem 5.1, using the left weak mixing of K, shows that the von Neumann algebras $B(L_{\text{even}}) \cap (A^{\text{op}})'$ and $B(L_{\text{odd}}) \cap (A^{\text{op}})'$ admit no N-central states that are normal on N. Note that we have the following decomposition of $L^2(z_0 M)$ as an N-A-bimodule:

$$L^2(z_0 M) = \Big(L_{\text{even}} \otimes_A \Big(L^2(A) \oplus \bigoplus_{n\geq 0} (H \ominus K)\, H^n\Big)\Big) \oplus \Big(L_{\text{odd}} \otimes_A \Big(L^2(A) \oplus \bigoplus_{n\geq 0} (H \ominus J(K))\, H^n\Big)\Big)\,.$$

This decomposition induces $*$-homomorphisms from $B(L_{\text{even}}) \cap (A^{\text{op}})'$ and $B(L_{\text{odd}}) \cap (A^{\text{op}})'$ to $B(z_0 L^2(M)) \cap (A^{\text{op}})' = z_0 \langle M, e_A\rangle z_0$. So, $z_0\langle M, e_A\rangle z_0$ admits no N-central state that is normal on N. A fortiori, (2) holds.

Next we define the projection $z_1 \in \mathcal{Z}(A)(1 - z_0)$ given by

$$z_1 = 1_{(1,+\infty]}\big(\Delta^\ell_{(1-z_0)H}\big)\,. \tag{6.2}$$

We also write $z = z_0 + z_1$ and $z_2 = 1 - z$.

Denote by $e' \in \mathcal{Z}(A)z_1$ the maximal projection with the following properties: the right support $f \in \mathcal{Z}(A)$ of $e'H$ satisfies $e'H = zHf$ and the A-bimodule $e'H$ is given by a partial automorphism of A. Define $e = z_1 - e'$.

By the definition of z_0, we get that the A-bimodule $(1 - z_0)H$ is a sum of A-bimodules that are finitely generated as a right Hilbert A-module. It then follows from the definition of z_1 that we can choose a projection $e_1 \in \mathcal{Z}(A)z_1$ that lies arbitrarily close to z_1 and for which there exists an A-subbimodule $L_1 \subset z_1 H$ with the following properties:

- the left support of L_1 equals e_1,
- L_1 is finitely generated as a right Hilbert A-module,
- $\Delta^\ell_{L_1}$ is bounded and satisfies $\Delta^\ell_{L_1} \geq \delta_1 e_1$ for some real number $\delta_1 > 1$.

Denote by e_2 the left support of $e_1(H \ominus L_1)$. Making e_1 slightly smaller, but still arbitrarily close to z_1, we may assume that e_2 is the left support of an A-subbimodule $L_2 \subset e_1(H \ominus L_1)$ with the following properties: L_2 is finitely generated as a right Hilbert A-module and $\Delta^\ell_{L_2}$ is bounded. By construction, $e_2 \leq e_1$. Since $e_2 L_1$ and L_2 are orthogonal and have the same left support e_2, it follows that for nonzero projections $s \in \mathcal{Z}(A)e_2$, the A-bimodule sH is not given by a partial automorphism of A. This means that $e_2 \leq e$ and thus, $e_2 \leq ee_1$. Define $L = L_1 + L_2$. Using notation (6.12), it follows from Lemma 6.5 that the left support of $e_2 L\, J(L) e_2 \cap (t_{e_2 L} A)^\perp$ equals e_2. A fortiori, the left support of $e_2 L\, Hz \cap (t_{e_2 L}A)^\perp$ equals e_2.

We put $e_3 = ee_1 - e_2$. Since e_2 is the left support of $e_1(H \ominus L_1)$, we get that $e_3 H = e_3 L_1 = e_3 L$. Since $e_3 \leq e$, applying Lemma 6.5 to the A-bimodule zH, we conclude that the left support of $e_3 L\, Hz \cap (t_{e_3 H} A)^\perp$ equals e_3. Summarizing, L has the following properties:

- the left support of L equals e_1,
- L is finitely generated as a right Hilbert A-module,
- Δ^ℓ_L is bounded and satisfies $\Delta^\ell_L \geq \delta e_1$ for some real number $\delta > 1$,

- the left support of $L\,Hz \cap (t_L A)^{\perp}$ equals ee_1.

Denote by $s \in \mathcal{Z}(A)$ the left support of $L\,H(z_0 + e_1) \cap (t_L A)^{\perp}$. Since e_1 could be chosen arbitrarily close to z_1, it follows that s lies arbitrarily close to e.

We next prove that

(4) $\mathcal{Z}(M)s \subset \mathcal{Z}(A)s$,

(5) every M-central state ω on $\langle M, e_A\rangle$ that is normal on M satisfies $\omega(s) = 0$.

Write $\Delta := \Delta_L^{\ell}$, choose a Pimsner-Popa basis $(\xi_i)_{i=1}^n$ for the right Hilbert A-module L and put

$$t := t_L = \sum_{i=1}^n \xi_i \otimes_A J(\xi_i) \; .$$

Since Δ is bounded, the vectors $\xi_i \in H$ are both left and right bounded.

Denoting by P_T the orthogonal projection onto a Hilbert subspace T, the main properties of t, used throughout the proof, are:

$$\langle t,t\rangle_A = {}_A\langle t,t\rangle = \Delta \quad , \quad \ell(\xi)^* t = J(P_L(\xi)) \;\text{ and }\; r(\xi)^* t = P_L(J(\xi)) \quad ,$$

for all left and right bounded vectors $\xi \in \mathcal{H}$.

Since the vectors ξ_i are both left and right bounded, we can define the self-adjoint element $S_1 \in e_1 M e_1$ given by

$$S_1 := \sum_{i=1}^n W(\xi_i, J(\xi_i)) \; .$$

By Lemma 6.8, the von Neumann algebra $D := \{S_1\}''$ is a subalgebra of $e_1 M e_1 \cap (Ae_1)'$ that is diffuse relative to Ae_1. We fix a unitary $u \in \mathcal{U}(D)$ satisfying $E_{Ae_1}(u^k) = 0$ for all $k \in \mathbb{Z}\setminus\{0\}$.

Defining

$$S_k := \sum_{i_1,\ldots,i_k=1}^n W(\xi_{i_1}, J(\xi_{i_1}), \ldots, \xi_{i_k}, J(\xi_{i_k})) \; ,$$

and denoting by $\Omega \in L^2(M)$ the vacuum vector, we get that

$$t_k := S_k\Omega = \underbrace{t \otimes_A \cdots \otimes_A t}_{k \text{ times}} \; . \tag{6.3}$$

With the convention that $S_0 = e_1$, the elements S_k, $k \geq 0$ span a dense $*$-subalgebra of D and are orthogonal in $L^2(D)$.

Proof of (4). We start by proving that an element $x \in \mathcal{Z}(M)e_1$ must be of a special form. Define the von Neumann subalgebra $E \subset e_1 M e_1$ given by $E := Ae_1 \vee D$. Define $T_0 \subset H^2$ as the closure of tA. Note that $\ell(t)\ell(t)^*\Delta^{-1}$ is the orthogonal projection of H^2 onto T_0. Then define $T_2 := H^2 \ominus T_0$ and $T_3 := H^3 \ominus (T_0 H + H T_0)$. Observe that $L^2(e_1 M e_1 \ominus E)$ is spanned by the D-subbimodules

$$\overline{DHD} \quad , \quad \overline{DT_2D} \quad , \quad \overline{DT_3D} \quad , \quad \overline{DT_2H^nT_2D} \;\text{ with } n \geq 0 \; . \tag{6.4}$$

Each of the D-bimodules in (6.4) is contained in a multiple of the coarse D-bimodule $L^2(D) \otimes L^2(D)$. This is only nontrivial for the first one $\overline{DHD}$. Fix a left and right bounded vector $\mu \in H$ with $\|\mu\| \leq 1$. Using the notation t_k introduced in (6.3), one checks that

$$S_k W(\mu)\Omega = t_k \otimes_A \mu + t_{k-1} \otimes_A P_L(\mu) \quad \text{and} \quad W(\mu) S_k \Omega = \mu \otimes_A t_k + P_{J(L)}(\mu) \otimes_A t_{k-1} \; .$$

When $\mu, \eta \in H$ are left and right bounded vectors, we have $\langle t_k \otimes_A \mu, \eta \otimes_A t_l \rangle = 0$ if $k \neq l$, while

$$\begin{aligned}\langle t_k \otimes_A \mu, \eta \otimes_A t_k \rangle &= \langle \ell(\eta)^*(t_k \otimes_A \mu), t_k \rangle \\ &= \langle J(P_L(\eta)) \otimes_A t_{k-1} \otimes_A \mu, t_k \rangle \\ &= \langle J(P_L(\eta)) \otimes_A t_{k-1}, r(\mu)^* t_k \rangle = \langle J(P_L(\eta)) \otimes_A t_{k-1}, t_{k-1} \otimes P_L(J(\mu)) \rangle \ .\end{aligned}$$

We can continue inductively and find complex numbers $\alpha_k, \beta_k, \gamma_k$ with modulus at most 1, depending on the vector μ that we keep fixed, such that

$$\langle S_k W(\mu) S_l, W(\mu) \rangle = \begin{cases} \alpha_k & \text{if } k = l \text{ and } k \geq 0, \\ \beta_{k-1} & \text{if } k = l+1 \text{ and } l \geq 0, \\ \gamma_k & \text{if } k = l-1 \text{ and } l \geq 1. \end{cases}$$

We next claim that

$$\xi := \sum_{k=0}^{\infty} \Big(\alpha_k (\Delta^{-k} S_k \otimes \Delta^{-k} S_k) + \beta_k (\Delta^{-k-1} S_{k+1} \otimes \Delta^{-k} S_k) + \gamma_k (\Delta^{-k} S_k \otimes \Delta^{-k-1} S_{k+1}) \Big)$$

is a well defined element in $L^2(E) \otimes L^2(E)$. This follows because $E_A(S_k^2) = \langle t_k, t_k \rangle_A = \Delta^k$ and thus

$$\|\Delta^{-k} S_k\|_2^2 = \tau(\Delta^{-2k} S_k^2) = \tau(\Delta^{-k}) \leq \delta^{-k} \ ,$$

where $\delta > 1$. By construction,

$$\langle S_k W(\mu) S_l, W(\mu) \rangle = \tau(e_1)^{-2} \, (\tau \otimes \tau)((S_k \otimes S_l)\xi) \ .$$

So, the D-bimodule $\overline{D\mu D}$ is contained in the coarse D-bimodule $L^2(E) \otimes L^2(E)$.

We have thus proved that all D-bimodules in (6.4) are contained in a multiple of the coarse D-bimodule. Since D is diffuse, it follows that $e_1 M e_1 \cap D' \subset E$. In particular, $\mathcal{Z}(M) e_1 \subset E$.

We are now ready to prove (4). Fix $x \in \mathcal{Z}(M)$. We have to prove that $xs \in A$. Because of (1) and the previous paragraphs, we can uniquely decompose $x(z_0 + e_1)$ as the $\| \cdot \|_2$-convergent sum

$$x(z_0 + e_1) = a_0 + \sum_{k=1}^{\infty} S_k a_k \tag{6.5}$$

with $a_0 \in A(z_0 + e_1)$ and $a_k \in A e_1$ for all $k \geq 1$. Note that $a_0 = E_A(x)(z_0 + e_1)$ and $a_k = \Delta^{-k} E_A(S_k x)$ for all $k \geq 1$.

Let now $\eta \in L\, H(z_0 + e_1) \cap (tA)^\perp$ be an arbitrary left and right bounded vector. Note that

$$\eta = \sum_{i=1}^{n} \xi_i \otimes_A J(\eta_i) \tag{6.6}$$

where the vectors $\eta_i \in (z_0 + e_1) H$ are both left and right bounded. Define

$$W(\eta) := \sum_{i=1}^{n} W(\xi_i, J(\eta_i))$$

and note that $W(\eta) \in sM(z_0 + e_1) \subset e_1 M(z_0 + e_1)$.

Using that $W(\eta)$ commutes with x and using the decomposition of $x(z_0+e_1)$ in (6.5), we find that

$$\begin{aligned} W(\eta)x\Omega = W(\eta)(z_0+e_1)x\Omega &= W(\eta)a_0\Omega + \sum_{k=1}^{\infty} W(\eta)S_k a_k \Omega \\ &= \eta(a_0+a_1) + \sum_{k=1}^{\infty} \eta \otimes_A t_k(a_k+a_{k+1}) \quad , \\ xW(\eta)\Omega = xe_1 W(\eta)\Omega &= a_0 e_1 W(\eta)\Omega + \sum_{k=1}^{\infty} a_k S_k W(\eta)\Omega \\ &= (a_0+a_1)\eta + \sum_{k=1}^{\infty} (a_k+a_{k+1}) t_k \otimes_A \eta \quad . \end{aligned}$$

In this last expression for $xW(\eta)\Omega$, all terms except $(a_0+a_1)\eta$ are orthogonal to $W(\eta)x\Omega$. We conclude that $(a_k+a_{k+1})t_k \otimes_A \eta = 0$ for all $k \geq 1$ and for all choices of η. Since the left support of $L\,H(z_0+e_1) \cap (tA)^\perp$ equals s, it follows that $(a_k+a_{k+1})s = 0$ for all $k \geq 1$. This means that $a_k s = (-1)^{k-1} a_1 s$ for all $k \geq 1$.

Since,

$$+\infty > \|x\|_2^2 \geq \sum_{k=1}^{\infty} \|S_k a_k s\|_2^2 = \sum_{k=1}^{\infty} \tau(s a_1^* \Delta^k a_1 s) \geq \sum_{k=1}^{\infty} \delta^k \|a_1 s\|_2^2 \; ,$$

it follows that $a_1 s = 0$. So, $a_k s = 0$ for all $k \geq 1$. From (6.5), it follows that $xs \in A$, so that (4) is proved.

Proof of (5). Fix an M-central state ω on $\langle M, e_A\rangle$ that is normal on M. We have to prove that $\omega(s) = 0$. Recall that we defined $T_0 \subset H^2$ as the closure of tA. Consider the following orthogonal decomposition of $e_1 L^2(M)$ as an A-bimodule:

$$\begin{aligned} & e_1 L^2(M) = V_0 \oplus V_1 \oplus V_2 \quad \text{where} \quad V_0 := \bigoplus_{n=0}^{\infty} T_0 H^n \quad , \\ & V_1 := L^2(Ae_1) \oplus \bigoplus_{n=0}^{\infty} (e_1 H \ominus L) H^n \quad , \quad V_2 := L \oplus \bigoplus_{n=0}^{\infty} (LH \ominus T_0) H^n \quad . \end{aligned}$$

Denote by $Q_i \in e_1 \langle M, e_A \rangle e_1$ the projections onto V_i, for $i = 0,1,2$. So, $e_1 = Q_0 + Q_1 + Q_2$. Also note that the projections Q_i commute with A. We prove below that $\omega(sQ_0) = \omega(Q_1) = \omega(Q_2) = 0$. Once these statements are proved, (5) follows.

To prove that $\omega(Q_1) = 0$, note that for all $\mu \in V_1$ and all $k \geq 1$, we have that $S_k \mu = t_k \otimes_A \mu$ and thus, $S_k\mu$ is orthogonal to V_1. So, for all $\mu, \mu' \in V_1$ and $d \in D$, we get that

$$\langle d\mu, \mu' \rangle = \tau(e_1)^{-1} \tau(d) \, \langle \mu, \mu' \rangle \; .$$

Above we introduced the unitary element $u \in \mathcal{U}(D)$ satisfying $\tau(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. It follows that the subspaces $u^k V_1$ are all orthogonal. So, the projections $u^k Q_1 u^{-k}$ are all orthogonal. By M-centrality, ω takes the same value on each of these projections. So, $\omega(Q_1) = 0$.

To prove that $\omega(Q_2) = 0$, we argue similarly. For all $\mu \in V_2$ and all $k \geq 2$, we have that $S_k \mu = t_k \otimes_A \mu + t_{k-1} \otimes_A \mu$ and thus, $S_k\mu$ is orthogonal to V_2. On the other hand, $S_1 \mu = t \otimes_A \mu + \mu$ and here, only $t \otimes_A \mu$ is orthogonal to V_2. It follows that for all $\mu, \mu' \in V_2$ and $d \in D$,

$$\langle d\mu, \mu' \rangle = \gamma(d) \, \langle \mu, \mu' \rangle \; ,$$

where $\gamma : D \to \mathbb{C}$ is the normal state given by $\gamma(e_1) = \gamma(S_1) = 1$ and $\gamma(S_k) = 0$ for all $k \geq 2$. Note that γ can be defined as well as the vector state on D implemented by any choice of unit vector in V_2. Since D is diffuse, we can choose a unitary $v \in \mathcal{U}(D)$ such that $\gamma(v^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. It follows that the subspaces $v^k V_2$ are all orthogonal. As in the previous paragraph, we get that $\omega(Q_2) = 0$.

It remains to prove that $\omega(sQ_0) = 0$. Fix $\eta \in L\, H(z_0 + e_1) \ominus T_0$ as in (6.6) and define

$$\eta' = \sum_{i=1}^{n} \eta_i \otimes_A J(\xi_i) \; .$$

Note that $\eta' \in (z_0 + e_1)H\, J(L) \ominus T_0$. From (2), we already know that $\omega(z_0) = 0$. Since $e_1\eta' \in V_1 + V_2$, we also know that $\omega(\ell(e_1\eta')\ell(e_1\eta')^*) = 0$. Both together imply that $\omega(\ell(\eta')\ell(\eta')^*) = 0$.

For all $n \geq 0$ and $\mu \in H^n$, we have that

$$W(\eta)(\eta' \otimes_A t \otimes_A \mu) = \eta \otimes_A \eta' \otimes_A t \otimes_A \mu + \sum_{i=1}^{n} \ell(\xi_i)\ell(\eta_i)^*(\eta' \otimes_A t \otimes_A \mu) + \langle \eta', \eta' \rangle_A \, (t \otimes_A \mu) \; .$$

Since

$$\ell(t)^* \sum_{i=1}^{n} \ell(\xi_i)\ell(\eta_i)^*\eta' = \sum_{i=1}^{n} \ell(J(\xi_i))^*\ell(\eta_i)^*\eta' = \ell(\eta')^*\eta' = \langle \eta', \eta' \rangle_A$$

and since the projection Q_0 is given by $Q_0 = \Delta^{-1}\ell(t)\ell(t)^*$, we get that

$$Q_0 W(\eta)(\eta' \otimes_A t \otimes_A \mu) = \langle \eta', \eta' \rangle_A \, \Delta^{-1} \, (t \otimes_A t \otimes_A \mu) + \langle \eta', \eta' \rangle_A \, (t \otimes_A \mu)$$

for all $n \geq 0$ and all $\mu \in H^n$. This means that

$$Q_0 W(\eta)\ell(\eta' \otimes_A t) = \langle \eta', \eta' \rangle_A \, \big(\Delta^{-1} \, \ell(t \otimes_A t) + \ell(t)\big) = \ell(t) \, \langle \eta', \eta' \rangle_A \, (1 + \Delta^{-1}\ell(t)) \; .$$

Because

$$\|\Delta^{-1}\ell(t)\|^2 = \|\Delta^{-2}\ell(t)^*\ell(t)\| = \|\Delta^{-1}\| \leq \delta^{-1} < 1 \; ,$$

the operator $R := 1 + \Delta^{-1}\ell(t)$ is invertible. Also note that there exists a $\kappa > 0$ such that

$$\ell(\eta' \otimes_A t)\ell(\eta' \otimes_A t)^* \leq \kappa \, \ell(\eta')\ell(\eta')^* \; .$$

So, we find $\varepsilon > 0$ and $\kappa > 0$ such that

$$\begin{aligned} \varepsilon \, \ell(t) \, (\langle \eta', \eta' \rangle_A)^2 \, \ell(t)^* &\leq \ell(t) \, \langle \eta', \eta' \rangle_A \, RR^* \, \langle \eta', \eta' \rangle_A \, \ell(t)^* \\ &= Q_0 W(\eta)\ell(\eta' \otimes_A t)\ell(\eta' \otimes_A t)^* W(\eta)^* Q_0 \\ &\leq \kappa Q_0 W(\eta)\ell(\eta')\ell(\eta')^* W(\eta)^* Q_0 \; . \end{aligned} \tag{6.7}$$

We already proved that $\omega(\ell(\eta')\ell(\eta')^*) = 0$. Since ω is M-central, also

$$\omega(W(\eta)\ell(\eta')\ell(\eta')^* W(\eta)^*) = 0 \; .$$

Because $e_1 = Q_0 + Q_1 + Q_2$ and $\omega(Q_1) = \omega(Q_2) = 0$, the Cauchy-Schwarz inequality implies that $\omega(Y) = \omega(Q_0 Y) = \omega(Y Q_0)$ for all $Y \in e_1\langle M, e_A\rangle e_1$. Therefore,

$$\omega\big(Q_0 W(\eta)\ell(\eta')\ell(\eta')^* W(\eta)^* Q_0\big) = \omega(W(\eta)\ell(\eta')\ell(\eta')^* W(\eta)^*) = 0 \; .$$

It then follows from (6.7) that

$$\omega\big((\langle \eta', \eta' \rangle_A)^2 \, \Delta \, Q_0\big) = 0$$

for all bounded vectors $\eta' \in (z_0 + e_1)H\, J(L) \ominus T_0$. By the Cauchy-Schwarz inequality and the normality of ω restricted to M, we get that $\omega(a_i Q_0) \to \omega(a Q_0)$ whenever $a_i \in A$ is a bounded sequence such that $\|a_i - a\|_2 \to 0$. Since the right support of the A-bimodule $(z_0+e_1)H\, J(L)\ominus T_0$ equals s, it follows that $\omega(sQ_0) = 0$. Since we already proved that $\omega(Q_1) = \omega(Q_2) = 0$, it follows that (5) holds.

Since s lies arbitrarily close to e, it follows from (1)-(2) and (4)-(5) that

(6) $\mathcal{Z}(M)(z_0 + e) \subset \mathcal{Z}(A)(z_0 + e)$,

(7) every M-central state ω on $\langle M, e_A\rangle$ that is normal on M satisfies $\omega(z_0 + e) = 0$.

Recall that $z = z_0+z_1$ and $z_2 = 1-(z_0+z_1)$. Note that $\Delta^\ell_{z_2 H} \leq z_2$. We claim that $z_2 H z_2 = \{0\}$. Denote by $e_0 \in \mathcal{Z}(A)z_2$ the left support of $z_2 H z_2$. Note that by symmetry, e_0 also is the right support of $z_2 H z_2$. By Lemma 6.7, we get that $\Delta^\ell_{e_0 H e_0} = e_0$ and that $e_0 H e_0$ is given by a partial automorphism of A. Since

$$\Delta^\ell_{e_0 H} = \Delta^\ell_{e_0 H e_0} + \Delta^\ell_{e_0 H(1-e_0)} = e_0 + \Delta^\ell_{e_0 H(1-e_0)}$$

and since $\Delta^\ell_{e_0 H} \leq e_0$, we get that $e_0 H(1 - e_0) = \{0\}$. We conclude that $e_0 H = H e_0 = e_0 H e_0$ and that this A-bimodule is given by a partial automorphism of A. Since H is assumed to be completely nontrivial, we get that $e_0 = 0$ and the claim is proved.

Recall that $e \in \mathcal{Z}(A)z_1$ was defined as $e = z_1 - e'$ where $e' \in \mathcal{Z}(A)z_1$ has the following properties: denoting by $f \in \mathcal{Z}(A)$ the right support of $e'H$, we have that $e'H = zHf$ and that the A-bimodule $e'H$ is given by a partial automorphism of A. We claim that $f \leq z$. To prove this claim, denote $f_1 := f z_2$. If $f_1 \neq 0$, we find a nonzero projection $e'' \in \mathcal{Z}(A)e'$ such that $e''H = zHf_1$ and such that this A-bimodule is given by a partial automorphism of A. Above, we have proved that $z_2 H z_2 = \{0\}$. A fortiori, $z_2 H f_1 = \{0\}$, meaning that $Hf_1 = zHf_1$. But then, $e''H = Hf_1$, contradicting the complete nontriviality of H. So, we have proved that $f \leq z$.

We next claim that $f \leq z_0 + e$. To prove this claim, assume that $f' := fe'$ is nonzero. Then, $f'H = fe'H = fzHf \subset Hz$ because $f \leq z$. Applying the symmetry J, it follows that $Hf' = zHf'$ and thus $e''H = Hf'$ for some nonzero projection $e'' \in \mathcal{Z}(A)e'$, again contradicting the complete nontriviality of H. So, we have proved that $f \leq z_0 + e$.

Since $e'H$ is given by a partial automorphism of A, we can take projections $e'' \in \mathcal{Z}(A)e'$ arbitrarily close to e' such that $e''H$ is finitely generated as a right Hilbert A-module and $\Delta^\ell_{e''H}$ is bounded. Denote by $f' \in \mathcal{Z}(A)f$ the right support of $e''H$ and denote by $\alpha : \mathcal{Z}(A)e'' \to \mathcal{Z}(A)f'$ the corresponding surjective $*$-isomorphism satisfying $a\xi = \xi\alpha(a)$ for all $a \in \mathcal{Z}(A)e''$. Let $(\gamma_i)_{i=1}^n$ be a Pimsner-Popa basis of the right A-module $e''H$ and define

$$R_i = \ell(\gamma_i) + \ell(J(\gamma_i))^* \quad \text{and} \quad R = \sum_{i=1}^n R_i R_i^* = \Delta^\ell_{e''H} + \sum_{i=1}^n W(\gamma_i, J(\gamma_i)) \; .$$

Note that $R_i \in e'' M f'$ and $R \in e'' M e''$. Since $\Delta^\ell_{e''H} = e''\Delta^\ell_H \geq e''$, it follows from Lemma 6.8 that the support projection of R equals e''.

Let $x \in \mathcal{Z}(M)$ and using (6), take $a \in \mathcal{Z}(A)(z_0 + e)$ such that $(z_0 + e)x = a$. Since $f' \leq z_0 + e$, we have $f'x = af'$ and thus

$$xR = \sum_{i=1}^n R_i x R_i^* = \sum_{i=1}^n R_i\, af'\, R_i^* = \alpha^{-1}(af')R \; .$$

Since the support projection of R equals e'', we have proved that $\mathcal{Z}(M)e'' \subset \mathcal{Z}(A)e''$. Since e'' lies arbitrarily close to e', together with (6), it follows that

(8) $\mathcal{Z}(M)z \subset \mathcal{Z}(A)z$.

A similar reasoning using (7) then implies that

(9) every M-central state ω on $\langle M, e_A\rangle$ that is normal on M satisfies $\omega(z) = 0$.

To prove the first two statements of the theorem, it remains to see what happens under the projection z_2.

Denote $\Delta_2 := \Delta^{\ell}_{z_2 H}$. By the definition of z_2, we have that $\Delta_2 \leq z_2$. Let $(\mu_i)_{i\in I}$ be a (possibly infinite) Pimsner-Popa basis for the right A-module $z_2 H$. Since Δ_2 is bounded, we may choose the vectors μ_i to be left and right bounded. For the same reason,

$$s := \sum_{i\in I} \mu_i \otimes_A J(\mu_i)$$

is a well defined bounded A-central vector in $z_2 H\, H z_2$ and the infinite sums

$$G_n = \sum_{i_1,\ldots,i_n} W(\mu_{i_1}, J(\mu_{i_1}), \ldots, \mu_{i_n}, J(\mu_{i_n}))$$

are well defined bounded operators in $z_2 M z_2 \cap (A z_2)'$ satisfying

$$G_n \Omega = s_n := \underbrace{s \otimes_A \cdots \otimes_A s}_{n \text{ times}} \; .$$

By convention, we put $G_0 = z_2$. From the definition of G_n, we obtain the recurrence relation

$$G_1 G_n = G_{n+1} + G_n + \Delta_2\, G_{n-1} \tag{6.8}$$

for all $n \geq 1$, and thus, $G_{n+1} = (G_1 - 1)G_n - \Delta_2\, G_{n-1}$ for all $n \geq 1$.

Denote by $q \in z_2 M z_2$ the projection onto the kernel of $G_1 + \Delta_2$. Although the sum defining G_1 is infinite, the computations in the proof of Lemma 6.8 remain valid and it follows that the kernel of $(G_1 + \Delta_2)\, 1_{\{1\}}(\Delta_2)$ is reduced to zero. So, $q \leq 1_{(0,1)}(\Delta_2)$.

With the convention that $s_0 = z_2\Omega$, we claim that

$$q\Omega = \sum_{k=0}^{\infty} (-1)^k (z_2 - \Delta_2) s_k = \sum_{k=0}^{\infty} (-1)^k s_k (z_2 - \Delta_2) \; . \tag{6.9}$$

Because

$$\begin{aligned}\sum_{k=0}^{\infty} \|(z_2 - \Delta_2)s_k\|_2^2 &= \sum_{k=0}^{\infty} \tau\bigl(\langle s_k, s_k\rangle_A\, (z_2 - \Delta_2)^2\bigr) \\ &= \sum_{k=0}^{\infty} \tau\bigl(\Delta_2^k (z_2 - \Delta_2)^2\bigr) = \tau(z_2 - \Delta_2) < \infty \; ,\end{aligned}$$

the right hand side of (6.9) is a well defined element $p \in L^2(z_2 M z_2)$ satisfying, with $\|\cdot\|_2$-convergence,

$$p = \sum_{k=0}^{\infty} (-1)^k (z_2 - \Delta_2)\, G_k \; .$$

Note that $p = p^*$. Using the recurrence relation (6.8), it follows that $(G_1 + \Delta_2)p = 0$ and thus $p = qp$. Taking the adjoint, also $p = pq$.

On the other hand, because $(G_1 + \Delta_2)q = 0$, we have $G_1 q = -\Delta_2 q$. Using the recurrence relation (6.8), it follows that $G_k q = (-1)^k \Delta_2^k q$ for all $k \geq 0$. It then follows that

$$pq = \sum_{k=0}^{\infty} (z_2 - \Delta_2)\Delta_2^k \, q = 1_{(0,1)}(\Delta_2) \, q = q \; .$$

We already proved that $pq = p$, so that $p = q$ and (6.9) is proved.

From (6.9), we get for all $\xi \in \mathcal{H}$ that

$$(\ell(\xi) + \ell(J(\xi))^*) \, q \, \Omega = (\ell(\xi z_2) + \ell(J(\xi z_2))^*) \, q \, \Omega = 0 \; .$$

So, for all $x \in M$, we have that $xq = E_A(x)q$. Taking the adjoint, also $qx = qE_A(x)$ for all $x \in M$. Since q commutes with A, it follows that $q \in \mathcal{Z}(M)$ and $Mq = Aq$. From (6.9), we also get that $E_A(q) = z_2 - \Delta_2$ and thus $E_A(q) = Z(\Delta_H^\ell)$ where $Z : (0, +\infty) \to \mathbb{R}$ is defined as in the formulation of the theorem. So, $E_A(1-q) = z + \Delta_2$ and this operator has support equal to 1. Statement (c) of the theorem is now proven.

We next prove that

(10) $\mathcal{Z}(M)(z_2 - q) \subset \mathcal{Z}(A)(z_2 - q)$.

Take $x \in \mathcal{Z}(M)$ and write

$$x z_2 \Omega = \sum_{n=0}^{\infty} \zeta_n \quad \text{with} \;\; \zeta_n \in z_2 H^n \; .$$

Using (8), take $a \in \mathcal{Z}(A)z$ such that $xz = a$. Also write $a_0 = E_A(xz_2)$ and note that $\zeta_0 = a_0 \Omega$. Since $z_2 \mathcal{H} z_2 = 0$, we have $z_2 \mathcal{H} = z_2 \mathcal{H} z$ and we get, for every $\xi \in \mathcal{H}$, that

$$\begin{aligned}\sum_{n=0}^{\infty} \bigl(\ell(\xi)^* + \ell(J(\xi))\bigr) \, \zeta_n &= \bigl(\ell(\xi)^* + \ell(J(\xi))\bigr) \, x z_2 \Omega \\ &= x \, \bigl(\ell(\xi)^* + \ell(J(\xi))\bigr) \, z_2 \Omega = x \, J(z_2 \xi) = xz \, J(z_2 \xi) = a \, J(z_2 \xi) \; .\end{aligned}$$

Comparing the components in H^n for all $n \geq 0$, we find that

$$\ell(\xi)^* \zeta_1 = 0 \quad , \quad \ell(\xi)^* \zeta_2 = a \, J(\xi) - J(\xi) \, a_0 \quad , \quad \ell(\xi)^* \zeta_{n+1} = -J(\xi) \otimes_A \zeta_{n-1}$$

for all $\xi \in z_2 \mathcal{H}$ and all $n \geq 2$. Since $\zeta_n \in z_2 H^n$ for all n, it first follows that $\zeta_1 = 0$ and then inductively, that $\zeta_n = 0$ for all odd n.

Next, we get that $\zeta_2 = s_a - s a_0$, where

$$s_a := \sum_{i \in I} \mu_i \otimes_A a J(\mu_i)$$

is a well defined A-central vector in $z_2 H^2 z_2$.

Before continuing the proof, we give another expression for s_a. For all $\mu, \mu' \in z_2 \mathcal{H} = z_2 \mathcal{H} z$, we have that $W(J(\mu), \mu') \in zMz$. Since $xz = a$ and $x \in \mathcal{Z}(M)$, it follows that a commutes with $W(J(\mu), \mu')$. This means that

$$a \, J(\mu) \otimes_A \mu' = J(\mu) \otimes_A \mu' \, a \quad \text{for all} \;\; \mu, \mu' \in z_2 \mathcal{H} \; .$$

It follows that $a \, J(\mu) \otimes_A s = J(\mu) \otimes_A s_a$ for all $\mu \in z_2 \mathcal{H}$. Defining the normal completely positive map $\varphi : Az \to Az_2$ given by

$$\varphi(b) = \sum_{i \in I} \langle J(\mu_i), b \, J(\mu_i) \rangle_A \quad \text{for all} \;\; b \in Az \; ,$$

we get that $\varphi(a)\, s = \Delta_2\, s_a$. Since $\varphi(z) = \Delta_2$, there is a unique normal completely positive map $\psi : Az \to Az_2$ such that $\psi(b)\Delta_2 = \varphi(b)$ for all $b \in Az$. We conclude that $s_a = \psi(a)\, s = s\, \psi(a)$.

Writing $a_1 = \psi(a) - a_0$, we get that $\zeta_2 = s\, a_1$. We then conclude that $\zeta_{2n} = (-1)^{n+1}\, s_n\, a_1$ for all $n \geq 1$. Define the spectral projection $r = 1_{\{1\}}(\Delta_2)$. Since

$$\langle \zeta_{2n}, \zeta_{2n} \rangle_A = a_1^* \, \langle s_n, s_n \rangle_A \, a_1 = a_1^* \, \Delta_2^n \, a_1 \; ,$$

we get that $\|\zeta_{2n} r\| = \|a_1 r\|_2$ for all n. Since $\sum_n \|\zeta_{2n} r\|^2 < \infty$, we conclude that $a_1 r = 0$ and thus $xr \in A$.

Using (6.9), it follows that $x(z_2 - \Delta_2) = qa_1 + a_2$ for some element $a_2 \in A$. Since $xr \in A$, it follows that $x(z_2 - q) \in A(z_2 - q)$. Since the support of $E_A(z_2 - q)$ equals z_2, it follows that (10) holds.

Using (8) and (10), to conclude the proof of statement (d), it suffices to prove that for any $a \in \mathcal{Z}(A)$, we have $a(1-q) \in \mathcal{Z}(M)$ if and only if $a \in C$, where C is defined in the formulation of the theorem. This follows immediately by expressing the commutation with $\ell(\xi) + \ell(J(\xi))^*$ for all $\xi \in \mathcal{H}$ and using that $\bigl(\ell(\xi) + \ell(J(\xi))^*\bigr)\, q = 0$, as shown above.

Let ω be an M-central state on $\langle M, e_A \rangle$ that is normal on M. To conclude the proof of statement (a), we have to show that $\omega(1 - q) = 0$. By (9), we already know that $\omega(z) = 0$. With $\mu_i \in z_2\mathcal{H} = z_2\mathcal{H}z$ as above, define $y_i := \ell(\mu_i) + \ell(J(\mu_i))^*$. Note that $y_i \in z_2 M z$ and that $G_1 + \Delta_2 = \sum_i y_i y_i^*$. By M-centrality and normality of ω on M, and because $y_i^* y_i \in zMz$, we get that $\omega(G_1 + \Delta_2) = 0$. So, $\omega(z_2 - q) = 0$. Since we already know that $\omega(z) = 0$, we conclude that $\omega(1 - q) = 0$.

It remains to prove statement (b). Assume that $s \in \mathcal{Z}(M)(1 - q)$ is a nonzero projection and that $B \subset Ms$ is a Cartan subalgebra. Since $\mathcal{N}_{Ms}(B)'' = Ms$, a combination of statement (a) and Theorem 4.1 implies that $B \prec_M A(1-q)$. The A-subbimodule $z_2 H = z_2 H z$ of $L^2(M)$ has finite right A-dimension equal to $\tau(\Delta_2)$ and realizes a full intertwining of $A(z_2 - q)$ into Az. It then follows that $B \prec_M Az$.

By [Po03, Theorem 2.1], we can take projections $q_1 \in B$, $p \in Az$, a faithful normal unital $*$-homomorphism $\theta : Bq_1 \to pAp$ and a nonzero partial isometry $v \in q_1 M p$ such that $bv = v\theta(b)$ for all $b \in Bq_1$. Since $B \subset Ms$ is maximal abelian, we may assume that $vv^* = q_1$. By [Io11, Lemma 1.5], we may assume that $B_0 := \theta(Bq_1)$ is a maximal abelian subalgebra of pAp. Write $q_2 = v^* v$ and note that $q_2 \in B_0' \cap pMp$. We may assume that the support projection of $E_A(q_2)$ equals p.

Since $z = z_0 + z_1$, at least one of the projections pz_0, pz_1 is nonzero. Since we can cut down everything with the projections z_0 and z_1, we may assume that either $p \leq z_0$ or $p \leq z_1$.

Proof in the case where $p \leq z_0$. Recall that we denoted by $K \subset H$ the largest A-subbimodule that is left weakly mixing and that z_0 is the left support of K. First assume that the B_0-A-bimodule pK is left weakly mixing. Define the orthogonal decomposition of the pAp-bimodule $pL^2(M)p$ given by

$$pL^2(M)p = U_1 \oplus U_2 \quad \text{with} \quad U_1 = \bigoplus_{n=0}^{\infty} pKH^n p \quad \text{and} \quad U_2 = L^2(pAp) \oplus \bigoplus_{n=0}^{\infty} p(H \ominus K)H^n p \; .$$

We claim that $v^* \mathcal{N}_{q_1 M q_1}(Bq_1) v \subset U_2$. To prove this claim, take $u \in \mathcal{N}_{q_1 M q_1}(Bq_1)$ and write $u^* b u = \alpha(b)$ for all $b \in Bq_1$. Put $x = v^* u v$ and denote by y the orthogonal projection of x onto U_1. Since U_1 is a pAp-subbimodule of $pL^2(M)p$, we get that y is a right pAp-bounded vector in U_1 and that $\theta(b) y = y \theta(\alpha(b))$ for all $b \in Bq$. Since the B_0-A-bimodule pK is left weakly mixing, also U_1 is left weakly mixing as a B_0-pAp-bimodule. So, we can take a sequence of unitaries $b_n \in \mathcal{U}(Bq_1)$ such that $\lim_n \|\langle \theta(b_n)y, y\rangle_{pAp}\|_2 = 0$. But,

$$\langle \theta(b_n)y, y\rangle_{pAp} = \langle y\theta(\alpha(b_n)), y\rangle_{pAp} = \theta(\alpha(b_n)^*)\,\langle y, y\rangle_{pAp} \; .$$

Since $\theta(\alpha(b_n))$ is a unitary in B_0, we have $\|\theta(\alpha(b_n)^*)\,\langle y, y\rangle_{pAp}\|_2 = \|\langle y, y\rangle_{pAp}\|_2$ for all n. We conclude that $y = 0$ and thus $v^*uv \in U_2$. Since the linear span of $\mathcal{N}_{q_1Mq_1}(Bq_1)$ is $\|\cdot\|_2$-dense in q_1Mq_1, we get that $q_2Mq_2 \subset U_2$.

Again consider the von Neumann subalgebra $N \subset z_0Mz_0$ introduced in (6.1). Since

$$P_{pL^2(N)p}(U_2) \subset L^2(pAp) \; ,$$

we get that $E_{pNp}(q_2Mq_2) \subset pAp$. Denote by $N_0 \subset pNp$ the von Neumann algebra generated by the subspace $E_{pNp}(q_2Mq_2)$. So, $N_0 \subset pAp$. In particular, $E_N(q_2) \in A$, so that $E_N(q_2) = E_A(q_2)$ and thus, $E_N(q_2)$ has support p. By [Io11, Lemma 1.6], the inclusion $N_0 \subset pNp$ is essentially of finite index in the sense of Definition 6.9. A fortiori, $pAp \subset pNp$ is essentially of finite index. This contradicts the left weak mixing of the N-A-bimodule $L^2(N)$ that we obtained in (3).

Next assume that the B_0-A-bimodule pK is not left weakly mixing and take a nonzero B_0-A-subbimodule $K_1 \subset pK$ that is finitely generated as a right Hilbert A-module. Denote by $z_0' \in \mathcal{Z}(B_0)$ the support projection of the left action of B_0 on K_1. Since $K_1 \neq \{0\}$, also $z_0' \neq 0$. Since the support of $E_A(q_2)$ equals p, we get that $E_A(q_2z_0') = E_A(q_2)z_0' \neq 0$. So, $q_2z_0' \neq 0$ and we can cut down everything by z_0' and assume that the left B_0 action on K_1 is faithful.

Put $P = \mathcal{N}_{pAp}(B_0)''$. Whenever $u \in \mathcal{N}_{q_1Mq_1}(Bq_1)$ with $ubu^* = \alpha(b)$ for all $b \in Bq_1$, we have $E_A(v^*uv)\theta(b) = \theta(\alpha(b))E_A(v^*uv)$ for all $b \in Bq_1$. Since $B_0 \subset pAp$ is maximal abelian, it follows that $E_A(v^*uv) \in P$. So $E_A(q_2Mq_2) \subset P$. From [Io11, Lemma 1.6], we conclude that the inclusion $P \subset pAp$ is essentially of finite index in the sense of Definition 6.9. So, all conditions of Lemma 6.10 are satisfied and we can choose a diffuse abelian von Neumann subalgebra $D \subset B_0' \cap pMp$ that is in tensor product position w.r.t. B_0. Since $Bq_1 \subset q_1Mq_1$ is maximal abelian, also $B_0q_2 \subset q_2Mq_2$ is maximal abelian. So, $q_2(B_0' \cap pMp)q_2 = B_0q_2$, contradicting Lemma 6.11 below.

Proof in the case where $p \leq z_1$. As proven above, we can find projections $e_1 \in \mathcal{Z}(A)z_1$ that lie arbitrarily close to z_1 and for which there exists an A-subbimodule $L \subset z_1H$ with the following properties: the left support of L equals e_1, L is finitely generated as a right Hilbert A-module, Δ_L^ℓ is bounded and $\Delta_L^\ell \geq e_1$. Taking e_1 close enough to z_1 and cutting down with e_1, we may assume that $p \leq e_1$. By Lemma 6.8, we can choose a diffuse abelian von Neumann subalgebra $D \subset (Ae_1)' \cap e_1Me_1$ that is in tensor product position w.r.t. Ae_1. Then $Dp \subset B_0' \cap pMp$ and Dp is in tensor product position w.r.t. B_0. Since Dp is diffuse abelian and $q_2 \in B_0' \cap pMp$ is a projection satisfying $q_2(B_0' \cap pMp)q_2 = B_0q_2$, this again contradicts Lemma 6.11. □

In the proof of Theorem 6.1, we needed several technical lemmas that we prove now.

Let (A, τ) be a tracial von Neumann algebra and denote by $\widehat{\mathcal{Z}(A)}$ the *extended positive part* of $\mathcal{Z}(A)$, i.e. when we identify $\mathcal{Z}(A) = L^\infty(X, \mu)$, then $\widehat{\mathcal{Z}(A)}$ consists of all measurable functions $f : X \to [0, +\infty]$ up to identification of functions that are equal almost everywhere.

Whenever (B, τ) and (A, τ) are tracial von Neumann algebras and H is a B-A-bimodule, we denote by $\Delta_H^\ell \in \widehat{\mathcal{Z}(B)}$ the unique element in the extended positive part of $\mathcal{Z}(B)$ characterized by

$$\tau(\Delta_H^\ell e) = \dim_{-A}(eH) \quad \text{for all projections} \;\; e \in \mathcal{Z}(B) \; . \tag{6.10}$$

Writing $H \cong p(\ell^2(\mathbb{N}) \otimes L^2(A))$ with the bimodule action given by $b \cdot \xi \cdot a = \alpha(b)\xi a$ where $\alpha : B \to p(B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} A)p$ is a normal $*$-homomorphism, we get that $\tau(\Delta_H^\ell \,\cdot\,) = (\operatorname{Tr} \otimes \tau)\alpha(\,\cdot\,)$ and this also allows to construct Δ_H^ℓ.

Recall that a *finitely generated* right Hilbert A-module K admits a *Pimsner-Popa basis*, i.e. right bounded elements $\xi_1, \dots, \xi_n$ such that

$$\xi = \sum_{i=1}^{n} \xi_i \, \langle \xi_i, \xi \rangle_A \tag{6.11}$$

for all right bounded elements $\xi \in K$. We denote by $t_K \in K \otimes_A \overline{K}$ the associated vector given by

$$t_K := \sum_{i=1}^{n} \xi_i \otimes_A \overline{\xi_i} \, . \tag{6.12}$$

When K is an A-bimodule, then t_K is an A-central vector and $\langle t_K, t_K \rangle_A = \Delta_K^\ell$.

Recall from the beginning of this section the notion of an A-bimodule given by a partial automorphism of A. Given an A-bimodule L, denote by $\operatorname{zdim}_{-A}(L)$, resp. $\operatorname{zdim}_{A-}(L)$, the center valued dimension of L as a right, resp. left A-module. These are elements in the extended positive part of $\mathcal{Z}(A)$. We have that L is finitely generated as a right Hilbert A-module if and only if $\operatorname{zdim}_{-A}(L)$ is bounded.

Lemma 6.5. *Let (A, τ) be a tracial von Neumann algebra and T an A-bimodule with left support e. Denote $\Sigma := \operatorname{zdim}_{A-}(T \otimes_A \overline{T})$. Then, the support of Σ equals e and $\Sigma \geq e$. Defining $e_1 = 1_{\{1\}}(\Sigma)$, the following holds.*

1. *Denoting by $f_1 \in \mathcal{Z}(A)$ the right support of $e_1 T$, we have that $e_1 T = T f_1$ and that the A-bimodule $e_1 T$ is given by a partial automorphism of A.*
2. *When $e_2 \in \mathcal{Z}(A)e$ and $f_2 \in \mathcal{Z}(A)$ are projections such that $e_2 T = T f_2$ and such that the A-bimodule $e_2 T$ is given by a partial automorphism of A, then $e_2 \leq e_1$.*
3. *If $e_0 \in \mathcal{Z}(A)e$ is a projection such that $e_0 T$ is finitely generated as a right Hilbert A-module, then the left support of $e_0 T \otimes_A \overline{T} \cap (t_{e_0 T} A)^\perp$ equals $e_0(1 - e_1)$.*

Proof. Choose a set I, a projection $p \in B(\ell^2(I)) \mathbin{\overline{\otimes}} A$ and a normal unital $*$-homomorphism $\alpha : A \to p(B(\ell^2(I)) \mathbin{\overline{\otimes}} A)p$ such that $T \cong p(\ell^2(I) \otimes L^2(A))$ with the A-bimodule structure given by $a \cdot \xi \cdot b = \alpha(a)\xi b$. Note that e equals the support of α. Also note that $T \otimes_A \overline{T} \cong L^2(p(B(\ell^2(I)) \mathbin{\overline{\otimes}} A)p)$ with the A-bimodule structure given by $a \cdot \xi \cdot b = \alpha(a)\xi\alpha(b)$.

Define $e_0 = 1_{(0,1]}(\Sigma)$ and denote by $f_0 \in \mathcal{Z}(A)$ the right support of $e_0 T$. Note that $(1 \otimes f_0)p$ is the central support of $\alpha(e_0)$ inside $p(B(\ell^2(I)) \mathbin{\overline{\otimes}} A)p$. By construction, $\operatorname{zdim}_{A-}(e_0 T \otimes_A \overline{T}) \leq e_0$. It follows that the commutant of the left A action on $e_0 T \otimes_A \overline{T}$ is a finite von Neumann algebra. A fortiori, $p(B(\ell^2(I)) \mathbin{\overline{\otimes}} A)p(1 \otimes f_0)$ is a finite von Neumann algebra. We can thus choose a sequence of projections $q_n \in \mathcal{Z}(A)f_0$ such that $q_n \to f_0$ and $p(1 \otimes q_n)$ has finite trace for all n. Denote by $p_n \in \mathcal{Z}(A)e_0$ the support of the homomorphism that maps $a \in Ae_0$ to $\alpha(a)(1 \otimes q_n)$. It follows that $p_n \to e_0$.

Since the closure of $\alpha(Ae_0)(1 \otimes q_n)$ inside $L^2(p(B(\ell^2(I)) \mathbin{\overline{\otimes}} A)p)$ has zdim_{A-} equal to p_n, we conclude that $\Sigma p_n \geq p_n$ for all n and thus $\Sigma e_0 \geq e_0$. From the definition of e_0, it then follows that $\Sigma e_0 = e_0$ and $e_0 = e_1$ (as defined in the formulation of the lemma), as well as $\Sigma \geq e$ and $f_0 = f_1$. Since $p_n \Sigma = p_n$ for all n, it also follows that $\alpha(Ap_n)(1 \otimes q_n)$ is dense in $\alpha(p_n)L^2(B(\ell^2(I)) \mathbin{\overline{\otimes}} A)p$ for all n, because the orthogonal complement has dimension zero.

This means that $\alpha(e_1) = (1 \otimes f_1)p$ and that $\alpha : Ae_1 \to p(B(\ell^2(I)) \overline{\otimes} A)p(1 \otimes f_1)$ is a surjective $*$-isomorphism. So, $e_1 T = T f_1$ and this A-bimodule is given by a partial automorphism of A.

So the first statement of the lemma is proved. Take $e_2 \in \mathcal{Z}(A)e$ and $f_2 \in \mathcal{Z}(A)$ as in the second statement of the lemma. It follows that $e_2 T \otimes_A \overline{T} = e_2 T \otimes_A \overline{e_2 T}$ and that $\operatorname{zdim}_{A-}(e_2 T \otimes_A \overline{T}) = e_2$. So, $e_2 \Sigma = e_2$, meaning that $e_2 \leq e_1$.

Finally take $e_0 \in \mathcal{Z}(A)$ as in the last statement of the lemma. We have $(\operatorname{Tr} \otimes \tau)\alpha(e_0) = \dim_{-A}(e_0 T) < \infty$. Under the above isomorphism between $T \otimes_A \overline{T}$ and $L^2(p(B(\ell^2(I)) \overline{\otimes} A)p)$, the vector $t_{e_0 T}$ corresponds to $\alpha(e_0)$. So we have to determine the left support z of $\alpha(e_0)pL^2(B(\ell^2(I)) \overline{\otimes} A)p \cap \alpha(Ae_0)^\perp$. A projection $e_3 \in \mathcal{Z}(A)e_0$ is orthogonal to z if and only if $\alpha(Ae_3)$ is dense in $\alpha(e_3)pL^2(B(\ell^2(I)) \overline{\otimes} A)p$. This holds if and only if there exists a projection $f_3 \in \mathcal{Z}(A)$ such that $\alpha(e_3) = (1 \otimes f_3)p$ and $\alpha(Ae_3) = p(B(\ell^2(I)) \overline{\otimes} A)p(1 \otimes f_3)$. Since this is equivalent with $e_3 \leq e_1$, we have proved that $z = e_0(1 - e_1)$. □

Lemma 6.6. *Let (A, τ) be a tracial von Neumann algebra and (H, J) a symmetric A-bimodule with left (and thus also, right) support $e \in \mathcal{Z}(A)$. There is a unique projection $e_1 \in \mathcal{Z}(A)$ such that $e_1 H = H e_1$, the A-bimodule $e_1 H$ is given by a partial automorphism of A and the $A(e - e_1)$-bimodule $(1 - e_1)H$ is completely nontrivial.*

Proof. By Lemma 6.5, we find projections $e_1, f_1 \in \mathcal{Z}(A)e$ such that $e_1 H = H f_1$, the A-bimodule $e_1 H$ is given by a partial automorphism of A and writing $e_2 := e - e_1$, $f_2 = e - f_1$, the Ae_2-Af_2-bimodule $e_2 H = H f_2$ is completely nontrivial. Since $H \cong \overline{H}$, we must have $e_1 = f_1$ and $e_2 = f_2$. The uniqueness of e_1 can be checked easily. □

By symmetry, given an A-bimodule H, we can also define $\Delta^r_H \in \widehat{\mathcal{Z}(A)}$ characterized by the formula $\tau(\Delta^r_H e) = \dim_{A-}(He)$ for every projection $e \in \mathcal{Z}(A)$.

Lemma 6.7. *Let (A, τ) be a tracial von Neumann algebra and T an A-bimodule with left support $e \in \mathcal{Z}(A)$ and right support $f \in \mathcal{Z}(A)$. If $\Delta^\ell_T \leq e$ and $\Delta^r_T \leq f$, then $\Delta^\ell_T = e$, $\Delta^r_T = f$ and T is given by a partial automorphism of A.*

Proof. Let $e_0 \in \mathcal{Z}(A)e$ be the maximal projection with the following properties: the right support $f_0 \in \mathcal{Z}(A)f$ of $e_0 T$ satisfies $e_0 T = T f_0$, the A-bimodule $e_0 T$ is given by a partial automorphism of A and $\Delta^\ell_T = e_0$, $\Delta^r_T = f_0$. We have to prove that $e_0 = e$.

Assume that e_0 is strictly smaller than e. Since $e_0 T = T f_0$, also f_0 is strictly smaller than f. Denote $e_1 = e - e_0$ and $f_1 = f - f_0$. Note that $e_1 T = T f_1$. Since $\dim_{-A}(T) = \tau(\Delta^\ell_T) \leq \tau(e) \leq 1$ and similarly $\dim_{A-}(T) \leq 1$, it follows from [PSV15, Proposition 2.3] that there exists a nonzero A-subbimodule $K \subset e_1 T$ with the following properties: K is finitely generated, both as a left Hilbert A-module and as a right Hilbert A-module, and denoting by $e_2 \in \mathcal{Z}(A)e_1$ and $f_2 \in \mathcal{Z}(A)f_1$ the left, resp. right, support of K, there is a surjective $*$-isomorphism $\alpha : \mathcal{Z}(A)f_2 \to \mathcal{Z}(A)e_2$ such that $\xi a = \alpha(a)\xi$ for all $\xi \in K$, $a \in \mathcal{Z}(A)f_2$.

Denote by D the Radon-Nikodym derivative between $\tau \circ \alpha$ and τ, so that $\tau(b) = \tau(\alpha(b)D)$ for all $b \in \mathcal{Z}(A)f_2$. By a direct computation, we get that

$$\Delta^\ell_K = D\, \alpha(\operatorname{zdim}_{-A}(K)) \quad \text{and} \quad \alpha(\Delta^r_K) = D^{-1} \operatorname{zdim}_{A-}(K) \; .$$

In particular, we get that

$$\Delta^\ell_K\, \alpha(\Delta^r_K) = \operatorname{zdim}_{A-}(K)\, \alpha(\operatorname{zdim}_{-A}(K)) \; . \tag{6.13}$$

By Lemma 6.5 and the computation in the proof of [PSV15, Lemma 2.2], we have

$$\operatorname{zdim}_{A-}(K)\, \alpha(\operatorname{zdim}_{-A}(K)) = \operatorname{zdim}_{A-}(K \otimes_A \overline{K}) \geq e_2 \; . \tag{6.14}$$

Since $\Delta^\ell_K \leq e_2$ and $\Delta^r_K \leq f_2$, in combination with (6.13), it follows that $\Delta^\ell_K = e_2$ and $\Delta^r_K = f_2$. From (6.14), we then also get that $\operatorname{zdim}_{A-}(K \otimes_A \overline{K}) = e_2$. By Lemma 6.5, K is given by a partial automorphism of A.

Since $e_2 \geq \Delta^\ell_{e_2T} = \Delta^\ell_K + \Delta^\ell_{e_2T\ominus K} = e_2 + \Delta^\ell_{e_2T\ominus K}$, we conclude that $e_2T \ominus K = \{0\}$. So, $e_2T = K$ and e_2T is given by a partial automorphism of A. This then contradicts the maximality of e_0. ☐

Lemma 6.8. *Let (A,τ) be a tracial von Neumann algebra and (H,J) a symmetric A-bimodule. Write $M = \Gamma(H,J,A,\tau)''$. Let $p \in A$ be a projection and $B \subset pAp$ a von Neumann subalgebra such that $B' \cap pAp = \mathcal{Z}(B)$. Let $K \subset pH$ be a B-A-subbimodule that is finitely generated as a right Hilbert A-module. Assume that Δ^ℓ_K is bounded and satisfies $\Delta^\ell_K \geq p$, as B-A-bimodule.*

Let $(\xi_k)_{k=1}^n$ be a Pimsner-Popa basis for K as a right A-module. Then the vectors ξ_k are also left A-bounded and using the notation of (3.2), we define $S \in pMp$ given by

$$S := \sum_{k=1}^n W(\xi_k, J(\xi_k)) \; . \tag{6.15}$$

Then, $S \in B' \cap pMp$, S is self-adjoint and S is diffuse relative to B. More precisely, in the von Neumann algebra $D := \{S\}''$, there exists a unitary $u \in \mathcal{U}(D)$ satisfying $E_B(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$.

Proof. Giving a Pimsner-Popa basis $(\xi_k)_{k=1}^n$ for the right Hilbert A-module K is the same as defining a right A-linear unitary operator $\theta : e(\mathbb{C}^n \otimes L^2(A)) \to K$ for some projection $e \in A^n := M_n(\mathbb{C}) \otimes A$, with $\xi_k = \theta(e(e_k \otimes 1))$. Define the faithful normal $*$-homomorphism $\alpha : B \to eA^n e$ such that $\theta(\alpha(b)\xi) = b\theta(\xi)$ for all $b \in B$ and $\xi \in e(\mathbb{C}^n \otimes L^2(A))$. View $\overline{\mathbb{C}^n} \otimes K$ as a B-A^n-subbimodule of $\overline{\mathbb{C}^n} \otimes pH$. Define the vector $\xi \in \overline{\mathbb{C}^n} \otimes K$ given by

$$\xi = \sum_{k=1}^n \overline{e_k} \otimes \xi_k \; .$$

Then, $b\xi = \xi\alpha(b)$ for all $b \in B$ and, in particular, $\xi \in (\overline{\mathbb{C}^n} \otimes K)e$.

Define the normal positive functional $\omega : pAp \to \mathbb{C} : \omega(a) = \langle a\xi, \xi\rangle$. Since ω is B-central and $B' \cap pAp = \mathcal{Z}(B)$, we find $\Delta \in L^1(\mathcal{Z}(B))^+$ such that $\omega(a) = \tau(a\Delta)$ for all $a \in pAp$. But for all projections $q \in B$, we have

$$\tau(q\Delta) = \omega(q) = \langle q\xi, \xi\rangle = \langle \xi\alpha(q), \xi\rangle = (\operatorname{Tr} \otimes \tau)(\alpha(q)) = \dim_{-A}(qK) \; .$$

This means that $\Delta = \Delta^\ell_K$. Since Δ^ℓ_K is bounded, the vectors $\xi_k \in H$ are left A-bounded.

So, the vectors ξ_k are both left and right A-bounded, so that the operator S given by (6.15) is a well defined element of pMp. Since

$$S = \sum_{k=1}^n \bigl(\ell(\xi_k)\ell(J(\xi_k)) + \ell(\xi_k)\ell(\xi_k)^* + \ell(J(\xi_k))^*\ell(\xi_k)^*\bigr) \; ,$$

we get that $S = S^*$. From this formula, we also get that S commutes with B. Put $S_1 := \Delta + S$. Since $\Delta \in \mathcal{Z}(B)$, it suffices to prove that S_1 is diffuse relative to B.

Write $A_1 = pAp$ and $A_2 = eA^n e$. Equip A_1 and A_2 with the non normalized traces given by restricting τ to A_1 and $\operatorname{Tr} \otimes \tau$ to A_2. View ξ as a vector in the A_1-A_2-bimodule $(\overline{\mathbb{C}^n} \otimes pH)e$ and note that

$$\langle \xi, \xi\rangle_{A_2} = e \quad , \quad {}_{A_1}\langle \xi, \xi\rangle = \Delta \; .$$

Denote $L := (\overline{\mathbb{C}^n} \otimes pH)e$. Recall that we view L as an A_1-A_2-bimodule and that $\xi \in L$. Write $L' := e(\mathbb{C}^n \otimes Hp)$, view L' as an A_2-A_1-bimodule and note that the anti-unitary operator

$$J_1 : L \to L' : J_1\big(\sum_{k=1}^{n} \overline{e_k} \otimes \mu_k\big) = \sum_{k=1}^{n} e_k \otimes J(\mu_k)$$

satisfies $J_1(a\mu b) = b^* J_1(\mu) a^*$ for all $\mu \in L$, $a \in A_1$ and $b \in A_2$. Define $\xi' \in L'$ given by $\xi' = J_1(\xi)\Delta^{-1/2}$. Then ξ' satisfies the following properties.

$$\langle \xi', \xi' \rangle_{A_1} = p \;\;,\;\; {}_{A_2}\langle \xi', \xi' \rangle = \alpha(\Delta^{-1}) \;\text{ and }\; \alpha(b)\xi' = \xi' b \;\;\forall b \in B \;.$$

Define the Hilbert spaces

$$\begin{aligned} L_{\text{even}} &= L^2(A_1) \oplus \bigoplus_{m=1}^{\infty} \big(L \otimes_{A_2} L'\big)^{\otimes_{A_1}^m} \;, \\ L_{\text{odd}} &= L' \otimes_{A_1} L_{\text{even}} = \bigoplus_{m=0}^{\infty} \Big(L' \otimes_{A_1} \big(L \otimes_{A_2} L'\big)^{\otimes_{A_1}^m}\Big) \;. \end{aligned}$$

Note that L_{even} is an A_1-bimodule, while L_{odd} is an A_2-A_1-bimodule. Then,

$$W := \ell(\xi')\Delta^{1/2} + \ell(\xi)^* \tag{6.16}$$

is a well defined bounded operator from L_{even} to L_{odd} and $W^*W \in B(L_{\text{even}})$.

Using the natural isometry $L \otimes_{A_2} L' \hookrightarrow p(H \otimes_A H)p$, we define the isometry $V : L_{\text{even}} \to pL^2(M)p$ given as the direct sum of the compositions of

$$\big(L \otimes_{A_2} L'\big)^{\otimes_{A_1}^m} \hookrightarrow \big(p(H \otimes_A H)p\big)^{\otimes_{A_1}^m} \hookrightarrow p\big(H^{\otimes_A^{2m}}\big)p \;.$$

Then V is A_1-bimodular and

$$V\, W^*W = S_1\, V \;. \tag{6.17}$$

To compute the $*$-distribution of $B \cup \{S_1\}$ w.r.t. the trace τ, it is thus sufficient to compute the $*$-distribution of $B \cup \{W^*W\}$ acting on L_{even} and w.r.t. the vector functional implemented by $p \in L^2(A_1) \subset L_{\text{even}}$.

Define the closed subspaces $L^0_{\text{even}} \subset L_{\text{even}}$ and $L^0_{\text{odd}} \subset L_{\text{odd}}$ given as the closed linear span

$$\begin{aligned} L^0_{\text{even}} &= \overline{\text{span}}\{L^2(B), (\xi \otimes_{A_2} \xi')^{\otimes_{A_1}^m} B \mid m \geq 1\} \;, \\ L^0_{\text{odd}} &= \overline{\text{span}}\{(\xi' \otimes_{A_1} (\xi \otimes_{A_2} \xi')^{\otimes_{A_1}^m})B \mid m \geq 0\} \;. \end{aligned}$$

Since $\xi \otimes_{A_2} \xi'$ is a B-central vector and since $\langle \xi, \xi \rangle_{A_2} = e$ and $\langle \xi', \xi' \rangle_{A_1} = p$, we find that $W(L^0_{\text{even}}) \subset L^0_{\text{odd}}$ and $W^*(L^0_{\text{odd}}) \subset L^0_{\text{even}}$. So to compute the $*$-distribution of $B \cup \{W^*W\}$, we may restrict B and W^*W to L^0_{even}.

Consider the full Fock space $\mathcal{F}(\mathbb{C}^2)$ of the 2-dimensional Hilbert space $\mathbb{C}^2$, with creation operators $\ell_1 = \ell(e_1)$ and $\ell_2 = \ell(e_2)$ given by the standard basis vectors $e_1, e_2 \in \mathbb{C}^2$. Denote by η the vector state on $B(\mathcal{F}(\mathbb{C}^2))$ implemented by the vacuum vector $\Omega \in \mathcal{F}(\mathbb{C}^2)$. For every $\lambda \geq 1$, consider the operator $X(\lambda) \in B(\mathcal{F}(\mathbb{C}^2))$ given by $X(\lambda) = \sqrt{\lambda}\ell_2 + \ell_1^*$. We find that $X(\lambda)^*X(\lambda) = \lambda y^* y$ with $y = \ell_2 + \lambda^{-1/2}\ell_1^*$. It then follows from [Sh96, Lemma 4.3 and discussion after Definition 4.1] that the spectral measure of $X(\lambda)^*X(\lambda)$ has no atoms. Also for every $\lambda \geq 1$, η is a faithful state on $\{X(\lambda)^*X(\lambda)\}''$.

Identify $\mathcal{Z}(B) = L^\infty(Z, \mu)$ for some standard probability space (Z, μ). View Δ as a bounded function from Z to $[1, +\infty)$ and define $Y \in B(\mathcal{F}(\mathbb{C}^2)) \mathbin{\overline{\otimes}} L^\infty(Z, \mu)$ given by $Y(z) = X(\Delta(z))$.

We can view Y as an element of $B(\mathcal{F}(\mathbb{C}^2)) \overline{\otimes} B$ acting on the Hilbert space $\mathcal{F}(\mathbb{C}^2) \otimes L^2(B)$. Also, $\eta \otimes \tau$ is faithful on $(1 \otimes B \cup \{Y^*Y\})''$. Define the isometry

$$U : L^0_{\text{even}} \to \mathcal{F}(\mathbb{C}^2) \otimes L^2(B) : U\big((\xi \otimes_{A_2} \xi')^{\otimes^m_{A_1}} b\big) = (e_1 \otimes e_2)^{\otimes m} \otimes b \; .$$

By construction, $UW^*W = Y^*YU$ and U is B-bimodular. It follows that the $*$-distribution of $B \cup \{S_1\}$ w.r.t. τ equals the $*$-distribution of $1 \otimes B \cup \{Y^*Y\}$ w.r.t. $\eta \otimes \tau$. So there is a unique normal $*$-isomorphism

$$\Psi : (1 \otimes B \cup \{Y^*Y\})'' \to (B \cup \{S_1\})''$$

satisfying $\Psi(1 \otimes b) = b$ for all $b \in B$ and $\Psi(Y^*Y) = S_1$. Also, $\tau \circ \Psi = \eta \otimes \tau$. Since for all $z \in Z$, the spectral measure of $Y(z)^*Y(z)$ has no atoms, there exists a unitary $v \in \{Y^*Y\}''$ such that $(\eta \otimes \tau)((1 \otimes b)v^k) = 0$ for all $b \in B$ and $k \in \mathbb{Z} \setminus \{0\}$. Taking $u = \Psi(v)$, the lemma is proved. $\square$

Definition 6.9 ([Va07, Definition A.2]). A von Neumann subalgebra P of a tracial von Neumann algebra (Q, τ) is said to be of *essentially finite index* if there exist projections $q \in P' \cap Q$ arbitrarily close to 1 such that $Pq \subset qQq$ has finite Jones index.

To make the connection with [Io11, Lemma 1.6], note that $P \subset Q$ is essentially of finite index if and only if $qQq \prec_{qQq} Pq$ for every nonzero projection $q \in P' \cap Q$.

Lemma 6.10. *Let (A, τ) be a tracial von Neumann algebra and (H, J) a symmetric A-bimodule. Write $M = \Gamma(H, J, A, \tau)''$.*

Let $p \in A$ be a projection and $B \subset pAp$ a von Neumann subalgebra such that $B' \cap pAp = \mathcal{Z}(B)$ and such that $\mathcal{N}_{pAp}(B)''$ has essentially finite index in pAp. Let $K_1 \subset pH$ be a B-A-subbimodule satisfying the following three properties.

1. *K_1 is a direct sum of B-A-subbimodules of finite right A-dimension.*
2. *The left action of B on K_1 is faithful.*
3. *The A-bimodule $\overline{AK_1}$ is left weakly mixing.*

Then there exists a diffuse abelian von Neumann subalgebra $D \subset B' \cap pMp$ that is in tensor product position w.r.t. B. More precisely, there exists a unitary $u \in B' \cap pMp$ such that $E_B(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$.

Proof. We claim that for every $\varepsilon > 0$, there exists a projection $z \in \mathcal{Z}(B)$ with $\tau(p - z) < \varepsilon$ and a B-A-subbimodule $L \subset zH$ such that L is finitely generated as a right Hilbert A-module and such that Δ^ℓ_L is bounded and satisfies $\Delta^\ell_L \geq z$. To prove this claim, denote $K := \overline{AK_1}$ and let $(K_i)_{i \in I}$ be a maximal family of mutually orthogonal nonzero B-A-subbimodules of pK that are finitely generated as a right A-module. Denote by R the closed linear span of all K_i. Whenever $u \in \mathcal{N}_{pAp}(B)$ and $i \in I$, also uK_i is a B-A-subbimodule of pK that is finitely generated as a right A-module. By the maximality of the family $(K_i)_{i \in I}$, we get that $uK_i \subset R$. So, $uR = R$ for all $u \in \mathcal{N}_{pAp}(B)$. Writing $P := \mathcal{N}_{pAp}(B)''$, we conclude that R is a P-A-subbimodule of pK.

Since $P \subset pAp$ is essentially of finite index and since ${}_A K_A$ is left weakly mixing, Lemma 6.12 says that for every projection $q \in P$, the right A-module qR is either $\{0\}$ or of infinite right A-dimension. By the assumptions of the lemma and the maximality of the family $(K_i)_{i \in I}$, the left B-action on R is faithful. So $qR \neq \{0\}$ and thus $\dim_{-A}(qR) = \infty$ for every nonzero projection $q \in B$. This means that for every nonzero projection $q \in B$,

$$\sum_{i \in I} \tau(q \Delta^\ell_{K_i}) = \sum_{i \in I} \dim_{-A}(qK_i) = \dim_{-A}(qR) = \infty \; .$$

So we can find a projection $z \in \mathcal{Z}(B)$ and a finite subset $I_0 \subset I$ such that $\tau(p-z) < \varepsilon$ and such that the operator $\Delta := \sum_{i \in I_0} \Delta^\ell_{K_i} z$ is bounded and satisfies $\Delta \geq z$. Defining $L = \sum_{i \in I_0} z K_i$, the claim is proved.

Combining the claim with Lemma 6.8, we find for every $\varepsilon > 0$, a projection $z \in \mathcal{Z}(B)$ with $\tau(p - z) < \varepsilon$ and a unitary $u \in (Bz)' \cap zMz$ such that $E_B(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. So, we find projections $z_n \in \mathcal{Z}(B)$ and unitaries $u_n \in (Bz_n)' \cap z_n M z_n$ such that $E_B(u_n^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$ and such that $\bigvee_n z_n = p$. We can then choose projections $z_n' \in \mathcal{Z}(B)$ with $z_n' \leq z_n$ and $\sum_n z_n' = p$. Defining $u = \sum_n u_n z_n'$, we have found a unitary in $B' \cap pMp$ satisfying $E_B(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. So, the lemma is proved.

□

Above we also needed the following two lemmas.

Lemma 6.11. *Let (N, τ) be a tracial von Neumann algebra and $B \subset N$ an abelian von Neumann subalgebra. Assume that $D \subset B' \cap N$ is a diffuse abelian von Neumann subalgebra that is in tensor product position w.r.t. B. Then there is no nonzero projection $q \in B' \cap N$ satisfying $q(B' \cap N)q = Bq$.*

Proof. Put $P = B' \cap N$ and assume that $q \in P$ is a nonzero projection such that $qPq = Bq$. Note that $B \subset \mathcal{Z}(P)$ because B is abelian. Take a nonzero projection $z \in \mathcal{Z}(P)$ such that $z = \sum_{i=1}^n v_i v_i^*$ where $v_1, \ldots, v_n$ are partial isometries in Pq. Note that $zq \neq 0$ and write $p = zq$. Then,

$$Pp = Pzq = zPq = \operatorname{span}\{v_i qPq \mid i = 1, \ldots, n\} = \operatorname{span}\{v_i B \mid i = 1, \ldots, n\} .$$

So, $L^2(P)p$ is finitely generated as a right Hilbert B-module. Define $Q = B \vee D$ and denote by $e \in Q$ the support projection of $E_Q(p)$. Then $\xi \mapsto \xi p$ is an injective right B-linear map from $L^2(Q)e$ to $L^2(P)p$. So also $L^2(Q)e$ is finitely generated as a right Hilbert B-module. Since $Q \cong B \overline{\otimes} D$ with D diffuse and since e is a nonzero projection in $Q \cong B \overline{\otimes} D$, this is absurd. □

Lemma 6.12. *Let (A, τ) be a tracial von Neumann algebra and ${}_A K_A$ an A-bimodule that is left weakly mixing. Let $p \in A$ be a projection and $P \subset pAp$ a von Neumann subalgebra that is essentially of finite index (see Definition 6.9). If $L \subset pK$ is a P-A-subbimodule and $q \in P$ is a projection such that $qL \neq \{0\}$, then the right A-dimension of qL is infinite.*

Proof. Assume for contradiction that $q \in P$ is a projection such that qL is nonzero and such that qL has finite right A-dimension.

Since $P \subset pAp$ is essentially of finite index, there exist projections $p_1 \in P' \cap pAp$ that lie arbitrarily close to p such that Ap_1 is finitely generated as a right Pp_1 module (purely algebraically using a Pimsner-Popa basis, see e.g. [Va07, A.2]). There also exist central projections $z \in \mathcal{Z}(P)$ that lie arbitrarily close to p such that Pzq is finitely generated as a right qPq-module. Take such p_1 and z with $p_1 zqL \neq \{0\}$. Then $Ap_1 zq$ is finitely generated as a right qPq-module. Therefore, the closed linear span of $Ap_1 zqL$ is a nonzero A-subbimodule of K having finite right A-dimension. This contradicts the left weak mixing of ${}_A H_A$. □

7 Compact groups, free subsets, c_0 probability measures and the proof of Theorem B

For every second countable compact group K with Haar probability measure μ and for every symmetric probability measure ν on K, we consider $A = L^\infty(K, \mu)$, the A-bimodule $H_\nu =$

$L^2(K \times K, \mu \times \nu)$ given by (1.1) and the symmetry $J_\nu : H_\nu \to H_\nu$ given by (1.2). We put $M = \Gamma(H_\nu, J_\nu, A, \mu)''$.

In Proposition 7.3 below, we characterize when the bimodule H_ν is mixing (so that M becomes strongly solid by Corollary 4.2) and when $A \subset M$ is an s-MASA. For the latter, the crucial property will be that the support S of ν is of the form $S = F \cup F^{-1}$ where $F \subset K$ is a closed subset that is *free* in the following sense.

Definition 7.1. A subset F of a group G is called *free* if

$$g_1^{\varepsilon_1} \cdots g_n^{\varepsilon_n} \neq e$$

for all nontrivial *reduced words*, i.e. for all $n \geq 1$ and all $g_1, \ldots, g_n \in F$, $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$ satisfying $\varepsilon_i = \varepsilon_{i+1}$ whenever $1 \leq i \leq n-1$ and $g_i = g_{i+1}$.

On the other hand, the mixing property of H_ν will follow from the following c_0 condition on the measure ν.

Whenever K is a compact group, we denote by $\lambda : K \to \mathcal{U}(L^2(K))$ the left regular representation. For every probability measure ν on K and every unitary representation $\pi : K \to \mathcal{U}(H)$, we denote

$$\pi(\nu) = \int_K \pi(x)\, d\nu(x) \; .$$

Definition 7.2. A probability measure ν on a compact group K is said to be c_0 if the operator $\lambda(\nu) \in B(L^2(K))$ is compact.

Note that ν is c_0 if and only if $\lambda(\nu)$ belongs to the reduced group C*-algebra $C_r^*(K)$. Also, since the regular representation of K decomposes as the direct sum of all irreducible representations of K, each appearing with multiplicity equal to its dimension, we get that a probability measure ν is c_0 if and only if

$$\lim_{\pi \in \mathrm{Irr}(K), \pi \to \infty} \|\pi(\nu)\| = 0 \; ,$$

i.e. if and only if the map $\mathrm{Irr}(K) \to \mathbb{R} : \pi \mapsto \|\pi(\nu)\|$ is c_0. In particular, when K is an abelian compact group, a probability measure ν on K is c_0 if and only if the Fourier transform of ν is a c_0 function on $\widehat{K}$.

Proposition 7.3. *Let K be a second countable compact group K with Haar probability measure μ. Put $A = L^\infty(K, \mu)$. Let ν be a symmetric probability measure on K without atoms. Define the A-bimodule H_ν with symmetry J_ν by (1.1) and (1.2). Denote by $M = \Gamma(H_\nu, J_\nu, A, \mu)''$ the associated tracial von Neumann algebra. Let S be the support of ν, i.e. the smallest closed subset of K with $\nu(S) = 1$.*

1. *The bimodule H_ν is weakly mixing, $A \subset M$ is a singular MASA, M has no Cartan subalgebra and $A \subset M$ is a maximal amenable subalgebra.*
2. *The von Neumann algebra M has no amenable direct summand. The center $\mathcal{Z}(M)$ of M equals $L^\infty(K/K_0)$ where $K_0 \subset K$ is the closure of the subgroup generated by S. So if S topologically generates K, then M is a nonamenable II_1 factor.*
3. *If S is of the form $S = F \cup F^{-1}$ where $F \subset K$ is a closed subset that is free in the sense of Definition 7.1, then $A \subset M$ is an s-MASA.*
4. *If ν is c_0 in the sense of Definition 7.2, then the bimodule H_ν is mixing. So then, M is strongly solid and whenever $B \subset M$ is an amenable von Neumann subalgebra for which $B \cap A$ is diffuse, we have $B \subset A$.*

Proof. 1. Note that

$$H_\nu^{\otimes_A^n} \cong L^2(K \times \underbrace{K \times \cdots \times K}_{n \text{ times}}, \mu \times \underbrace{\nu \times \cdots \times \nu}_{n \text{ times}}) \tag{7.1}$$

with the A-bimodule structure given by

$$(F \cdot \xi \cdot G)(x, y_1, \ldots, y_n) = F(xy_1 \cdots y_n)\, \xi(x, y_1, \ldots, y_n)\, G(x) \; .$$

Define $D \subset K \times K$ given by $D = \{(y, y^{-1}) \mid y \in K\}$. Since ν has no atoms, we have $(\nu \times \nu)(D) = 0$. It then follows that $H_\nu \otimes_A H_\nu$ has no nonzero A-central vectors. By Proposition 2.3, the A-bimodule H_ν is weakly mixing. So also $L^2(M) \ominus L^2(A)$ is a weakly mixing A-bimodule, implying that $\mathcal{N}_M(A) \subset A$. So, $A \subset M$ is a MASA and this MASA is singular. By Theorem 6.1, M has no Cartan subalgebra. By Theorem 5.1, we get that $A \subset M$ is a maximal amenable subalgebra.

2. Since H_ν is weakly mixing, we get from Theorem 5.1 that M has no amenable direct summand and that $\mathcal{Z}(M)$ consists of all $a \in A$ satisfying $a \cdot \xi = \xi \cdot a$ for all $\xi \in H_\nu$. It is then clear that $L^\infty(K/K_0) \subset \mathcal{Z}(M)$. To prove the converse, fix $a \in A$ with $a \cdot \xi = \xi \cdot a$ for all $\xi \in H_\nu$. We find in particular that $a(xy) = a(x)$ for $\mu \times \nu$-a.e. $(x, y) \in K \times K$. Let $\mathcal{U}_n$ be a decreasing sequence of basic neighborhoods of e in K. Define the functions b_n given by

$$b_n(y) = \mu(\mathcal{U}_n)^{-1} \int_{\mathcal{U}_n} a(xy)\, d\mu(x) \; .$$

For every fixed n, the functions b_n still satisfy $b_n(xy) = b(x)$ for $\mu \times \nu$-a.e. $(x, y) \in K \times K$. But the functions b_n are continuous. It follows that $b_n(xy) = b_n(x)$ for all $x \in K$ and all $y \in S$. So, $b_n \in C(K/K_0)$. Since $\lim_n \|b_n - a\|_1 = 0$, we get that $a \in L^\infty(K/K_0)$.

3. Denote by $W_n \subset (F \cup F^{-1})^n$ the subset of reduced words of length n. Since ν has no atoms, we find that $\nu^n(W_n) = 1$. Denote by $\pi_n : K^n \to K$ the multiplication map and put $S_n := \pi_n(W_n)$. Since F is free, the subsets $S_n \subset K$ are disjoint. By freeness of F, we also have that the restriction of π_n to W_n is injective. Define the probability measures $\nu_n := (\pi_n)_*(\nu^n)$ and then $\eta = \frac{1}{2}\delta_0 + \sum_{n=1}^\infty 2^{-n-1}\nu_n$. Using (7.1), it follows that ${}_A L^2(M)_A$ is isomorphic with the A-bimodule

$$L^2(K \times K, \mu \times \eta) \quad \text{with} \quad (F \cdot \xi \cdot G)(x, y) = F(xy)\, \xi(x, y)\, G(x) \; .$$

So, ${}_A L^2(M)_A$ is a cyclic bimodule and $A \subset M$ is an s-MASA.

4. Define $\xi_0 \in H_\nu$ by $\xi_0(x, y) = 1$ for all $x, y \in K$. Denote by $\varphi : A \to A$ the completely positive map given by $\varphi(a) = \langle \xi_0, a\xi_0 \rangle_A$. To prove that H_ν is mixing, it is sufficient to prove that $\lim_n \|\varphi(a_n)\|_2 = 0$ whenever (a_n) is a bounded sequence in A that converges weakly to 0. Denoting by $\rho : K \to L^2(K)$ the right regular representation, we get that $\varphi(a) = \rho(\nu)(a)$ for all $a \in A \subset L^2(K)$. Since $\rho(\nu)$ is a compact operator, we indeed get that $\lim_n \|\rho(\nu)(a_n)\|_2 = 0$. So, H_ν is a mixing A-bimodule. By Corollary 4.2, M is strongly solid. The remaining statement follows from Theorem 5.1. ☐

Remark 7.4. In the special case where K is abelian, we identify $L^\infty(K, \mu) = L(G)$, with $G := \widehat{K}$ being a countable abelian group. Then the symmetric $L^\infty(K, \mu)$-bimodule H_ν given by (1.1) and (1.2) is isomorphic with the symmetric $L(G)$-bimodule associated, as in Remark 3.5, with the cyclic orthogonal representation of G with spectral measure ν. In particular, as in Remark 3.5, the von Neumann algebras $M = \Gamma(H_\nu, J_\nu, L^\infty(K), \mu)''$ can also be realized as a free Bogoljubov crossed product by the countable abelian group G. In this way, Proposition 7.3 generalizes the results of [HS09, Ho12a]. Note however that for a free Bogoljubov crossed product $M = \Gamma(K_\mathbb{R})'' \rtimes G$ with G abelian, the subalgebra $L(G) \subset M$ is *never* an s-MASA. So our more general construction is essential to prove Theorem B.

For non abelian compact groups K, we can still view $K = \widehat{G}$, but G is no longer a countable group, rather a discrete Kac algebra. It is then still possible to identify the II_1 factors M in Proposition 7.3 with a crossed product $\Gamma(K_\mathbb{R})'' \rtimes G$, where the discrete Kac algebra action of G on $\Gamma(K_\mathbb{R})''$ is the free Bogoljubov action associated in [Va02] with an orthogonal corepresentation of the quantum group G.

The main result of this section says that in certain sufficiently non abelian compact groups K, one can find "large" free subsets $F \subset K$, where "large" means that F carries a non atomic probability measure that is c_0. We conjecture that the compact Lie groups $\mathrm{SO}(n)$, $n \geq 3$, admit free subsets carrying a c_0 probability measure. For our purposes, it is however sufficient to prove that these exist in more ad hoc groups.

For every prime number p, denote by Γ_p the finite group $\Gamma_p = \mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$. The following is the main result of this section. Recall that the support of a probability measure ν on a compact space K is defined as the smallest closed subset $S \subset K$ with $\nu(S) = 1$.

Theorem 7.5. *There exists a sequence of prime numbers p_n tending to infinity, a closed free subset $F \subset K := \prod_{n=1}^\infty \Gamma_{p_n}$ topologically generating K and a symmetric, non atomic, c_0 probability measure ν on K whose support equals $F \cup F^{-1}$.*

We then immediately get:

Proof of Theorem B. Take K and ν as in Theorem 7.5. Denote by M the associated von Neumann algebra with abelian subalgebra $A \subset M$ as in Proposition 7.3. By Proposition 7.3, we get that M is a nonamenable, strongly solid II_1 factor and that $A \subset M$ is an s-MASA. $\square$

Before proving Theorem 7.5, we need some preparation.

The Alon-Roichman theorem [AR92] asserts that the Cayley graph given by a random and independent choice of $k \geq c(\varepsilon) \log |G|$ elements in a finite group G has expected second eigenvalue at most ε, with the normalization chosen so that the largest eigenvalue is 1. In [LR04, Theorem 2], a simple proof of that result was given. The same proofs yields the following result. For completeness, we provide the argument.

Whenever G is a group, $\pi : G \to \mathcal{U}(H)$ is a unitary representation and $g_1, \ldots, g_k \in G$, we write

$$\pi(g_1, \ldots, g_k) := \frac{1}{k} \sum_{j=1}^k \pi(g_j) \; . \tag{7.2}$$

Lemma 7.6 ([LR04])**.** *Let G_n be a sequence of finite groups and k_n a sequence of positive integers such that $k_n / \log |G_n| \to \infty$. For every $\varepsilon > 0$ and for a uniform and independent choice of k_n elements $g_1, \ldots, g_{k_n} \in G_n$, we have that*

$$\lim_{n\to\infty} \mathrm{P}\Big(\; \|\pi(g_1, \ldots, g_{k_n})\| \leq \varepsilon \;\; \textit{for all} \;\; \pi \in \mathrm{Irr}(G_n) \setminus \{\epsilon\} \; \Big) = 1 \; .$$

Proof. Fix a finite group G and a positive integer k. Let $g_1, \ldots, g_k$ be a uniform and independent choice of elements of G. Denote by $\lambda_0 : G \to \mathcal{U}(\ell^2(G) \ominus \mathbb{C}1)$ the regular representation restricted to $\ell^2(G) \ominus \mathbb{C}1$. Put $d = |G| - 1$. Both

$$T(g_1, \ldots, g_k) = \frac{1}{k} \sum_{j=1}^k \frac{\lambda_0(g_j) + \lambda_0(g_j)^*}{2} \quad \text{and} \quad S(g_1, \ldots, g_k) = \frac{1}{k} \sum_{j=1}^k \frac{i\lambda_0(g_j) - i\lambda_0(g_j)^*}{2}$$

are sums of k independent self-adjoint $d \times d$ matrices of norm at most 1 and having expectation 0. We apply [AW01, Theorem 19] to the independent random variables

$$X_j = \frac{2 + \lambda_0(g_j) + \lambda_0(g_j)^*}{4} \; ,$$

satisfying $0 \leq X_j \leq 1$ and having expectation $1/2$. We conclude that for every $0 \leq \varepsilon \leq 1/2$,

$$\mathrm{P}\big(\|T(g_1, \ldots, g_k)\| \leq \varepsilon \big) = \mathrm{P}\Big((1-\varepsilon)\frac{1}{2} \leq \frac{1}{k} \sum_{j=1}^{k} X_j \leq (1+\varepsilon)\frac{1}{2} \Big) \geq 1 - 2d \, \exp\big(-k \frac{\varepsilon^2}{4 \log 2}\big) \; .$$

The same estimate holds for $S(g_1, \ldots, g_k)$. Since $\lambda_0(g_1, \ldots, g_k) = T(g_1, \ldots, g_k) - iS(g_1, \ldots, g_k)$ and since λ_0 is the direct sum of all nontrivial irreducible representations of G (all appearing with multiplicity equal to their dimension), we conclude that

$$\mathrm{P}\big(\|\pi(g_1, \ldots, g_k)\| \leq \varepsilon \;\text{ for all }\; \pi \in \mathrm{Irr}(G) \setminus \{\epsilon\} \big) \geq 1 - 4|G| \, \exp\big(-k \frac{\varepsilon^2}{16 \log 2}\big) \; .$$

Taking $G = G_n$, $k = k_n$ and $n \to \infty$, our assumption that $k_n / \log |G_n| \to \infty$ implies that for every fixed $\varepsilon > 0$,

$$|G_n| \, \exp\big(-k_n \frac{\varepsilon^2}{16 \log 2}\big) \to 0$$

and thus the lemma follows. □

On the other hand in [GHSSV07], it is proven that random Cayley graphs of the groups $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ have large girth. More precisely, we say that elements $g_1, \ldots, g_k$ in a group G satisfy no relation of length $\leq \ell$ if every nontrivial reduced word of length at most ℓ with letters from $g_1^{\pm 1}, \ldots, g_k^{\pm 1}$ defines a nontrivial element in G.

The estimates in the proof of [GHSSV07, Lemma 10] give the following result. Again for completeness, we provide the argument.

Lemma 7.7 ([GHSSV07])**.** *Let p_n be a sequence of prime numbers tending to infinity and let k_n be a sequence of positive integers such that $\log k_n / \log p_n \to 0$. Put $\Gamma_{p_n} = \mathrm{PGL}_2(\mathbb{Z}/p_n\mathbb{Z})$. For every $\ell > 0$ and for a uniform and independent choice of k_n elements $g_1, \ldots, g_{k_n} \in \Gamma_{p_n}$, we have that*

$$\lim_{n \to \infty} \mathrm{P}\Big(g_1, \ldots, g_{k_n} \;\text{ satisfy no relation of length } \leq \ell \Big) = 1 \; .$$

Proof. Let G be a group. A *law* of length ℓ in G is a nontrivial element w in a free group $\mathbb{F}_n$ such that w has length ℓ and $w(g_1, \ldots, g_n) = e$ for all $g_1, \ldots, g_n \in G$. For example, if G is abelian, the element $w = aba^{-1}b^{-1}$ of $\mathbb{F}_2$ defines a law of length 4 in G. Since the labeling of the generators does not matter, any law of length ℓ can be defined by a nontrivial element of $\mathbb{F}_n$ with $n \leq \ell$. In particular, there are only finitely many possible laws of a certain length ℓ.

Since $\mathbb{F}_\infty \hookrightarrow \mathbb{F}_2 \hookrightarrow \mathrm{PSL}_2(\mathbb{Z})$, the group $\mathrm{PSL}_2(\mathbb{Z})$ satisfies no law. For every prime number p, write $\Gamma_p = \mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$. Using the quotient maps $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$, we get that a given nontrivial element $w \in \mathbb{F}_n$ can be a law for at most finitely many Γ_p. So, for every $\ell > 0$, we get that Γ_p satisfies no law of length $\leq \ell$ for all large enough primes p. (Note that [GHSSV07, Proposition 11] provides a much more precise result.)

Let $w = g_{i_1}^{\varepsilon_1} \cdots g_{i_\ell}^{\varepsilon_\ell}$ with $i_j \in \{1, \ldots, k\}$ and $\varepsilon_j \in \{\pm 1\}$ be a reduced word of length ℓ in $g_1^{\pm 1}, \ldots, g_k^{\pm 1}$. Let p be a prime number and assume that w is not a law of Γ_p. With the same argument as in the proof of [GHSSV07, Lemma 10], we now prove that for a uniform and independent choice of $g_1, \ldots, g_k \in \Gamma_p$, we have that

$$\mathrm{P}\big(w(g_1, \ldots, g_k) = e \text{ in } \Gamma_p \big) \leq \frac{\ell}{p} \big(1 + \frac{1}{p-1}\big)^{3k} . \tag{7.3}$$

Denote $F_p = \mathbb{Z}/p\mathbb{Z}$, not to be confused with the free group $\mathbb{F}_p$. Write $G_p = GL_2(F_p) \subset F_p^{2\times 2}$. Define the map

$$W : \big(F_p^{2\times 2}\big)^k \to F_p^{2\times 2} : W(a_1, \ldots, a_k) = b_{i_1} \cdots b_{i_\ell}$$

where $b_{i_j} = a_{i_j}$ when $\varepsilon_j = 1$ and b_{i_j} equals the adjunct matrix of a_{i_j} when $\varepsilon_j = -1$. Note that the four components W_{st}, $s, t \in \{1, 2\}$, of the map W are polynomials of degree at most ℓ in the $4k$ variables $a \in \big(F_p^{2\times 2}\big)^k$. Define the subset $\mathcal{W} \subset \big(F_p^{2\times 2}\big)^k$ given by

$$\begin{aligned} \mathcal{W} &= \big\{ a \in \big(F_p^{2\times 2}\big)^k \mid W(a) \text{ is a multiple of the identity matrix} \big\} \\ &= \big\{ a \in \big(F_p^{2\times 2}\big)^k \mid W_{11}(a) - W_{22}(a) = W_{12}(a) = W_{21}(a) = 0 \big\} . \end{aligned}$$

We also define $\mathcal{V} = \mathcal{W} \cap (G_p)^k$ and

$$\mathcal{U} = \{ g \in (\Gamma_p)^k \mid w(g_1, \ldots, g_k) = e \text{ in } \Gamma_p \} .$$

The quotient map $G_p \to \Gamma_p$ induces the $(p-1)^k$-fold covering $\pi : \mathcal{V} \to \mathcal{U}$.

The subset $\mathcal{W} \subset F_p^{4k}$ is the solution set of a system of three polynomial equations of degree at most ℓ. If each of these polynomials is identically zero, we get that $\mathcal{W} = F_p^{4k}$ and thus $\mathcal{U} = (\Gamma_p)^k$. This means that w is a law of Γ_p, which we supposed not to be the case. So at least one of the polynomials is not identically zero. The number of zeros of such a polynomial is bounded above by ℓp^{4k-1} (and a better, even optimal, bound can be found in [Se89]). So, $|\mathcal{W}| \leq \ell p^{4k-1}$. Then also $|\mathcal{V}| \leq \ell p^{4k-1}$ and because π is a $(p-1)^k$-fold covering, we find that

$$|\mathcal{U}| \leq \ell \, (p-1)^{-k} \, p^{4k-1} .$$

Since $|\Gamma_p| = (p-1)\, p \,(p+1)$, we conclude that

$$\mathrm{P}\big(w(g_1, \ldots, g_k) = e \text{ in } \Gamma_p \big) = \frac{|\mathcal{U}|}{|\Gamma_p|^k} \leq \frac{\ell}{p} \,(p-1)^{-2k} \,(p+1)^{-k} \, p^{3k} \leq \frac{\ell}{p} \big(1 + \frac{1}{p-1}\big)^{3k} .$$

So, (7.3) holds.

Now assume that p_n is a sequence of prime numbers and k_n are positive integers such that $p_n \to \infty$ and $\log k_n / \log p_n \to 0$. For all n large enough, $3k_n \leq p_n - 1$ and for all n large enough, as we explained in the beginning of the proof, Γ_{p_n} has no law of length $\leq \ell$. Since $(1 + 1/x)^x < 3$ for all $x > 0$ and since there are less than $(2k)^{\ell+1}$ reduced words of length $\leq l$ in $g_1^{\pm 1}, \ldots, g_k^{\pm 1}$, we find that for all n large enough and a uniform, independent choice of $g_1, \ldots, g_{k_n} \in \Gamma_{p_n}$, we have

$$\mathrm{P}\Big(\; g_1, \ldots, g_{k_n} \text{ satisfy a relation of length} \leq \ell \text{ in } \Gamma_{p_n} \Big) \leq (2k_n)^{\ell+1} \, \frac{3\ell}{p_n} .$$

By our assumption that $\log k_n / \log p_n \to 0$, the right hand side tends to 0 as $n \to \infty$ and the lemma is proved. ☐

Combining Lemmas 7.6 and 7.7, we obtain the following.

Lemma 7.8. *For all $\varepsilon > 0$ and all $k_0, p_0, \ell \in \mathbb{N}$, there exist a prime number $p \geq p_0$, an integer $k \geq k_0$ and elements $g_1, \ldots, g_k \in \Gamma_p = \mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ generating the group Γ_p such that*

1. $\|\pi(g_1,\ldots,g_k)\| \leq \varepsilon$ *for every nontrivial irreducible representation* $\pi \in \mathrm{Irr}(\Gamma_p)$,

2. $g_1,\ldots,g_k$ *satisfy no relation of length* $\leq \ell$.

Proof. Choose any sequence of prime numbers p_n tending to infinity. Define $k_n = \lfloor (\log p_n)^2 \rfloor$. Since $|\Gamma_{p_n}| = (p_n - 1)\, p_n\, (p_n + 1)$, we get that $k_n / \log |\Gamma_{p_n}| \to \infty$. Also, $\log k_n / \log p_n \to 0$. So Lemmas 7.6 and 7.7 apply and for a large enough choice of n, properties 1 and 2 in the lemma hold for $p = p_n$, $k = k_n$ and a large portion of the k_n-tuples $(g_1,\ldots,g_{k_n}) \in \Gamma_{p_n}^{k_n}$.

The first property in the lemma is equivalent with

$$\Big\| \Big(\frac{1}{k} \sum_{j=1}^{k} \lambda(g_j) \Big)_{\ell^2(\Gamma_p)\ominus\mathbb{C}1} \Big\| \leq \varepsilon \; ,$$

where $\lambda : \Gamma_p \to \ell^2(\Gamma_p)$ is the regular representation. If $\varepsilon < 1$, it then follows in particular that there are no non zero functions in $\ell^2(\Gamma_p) \ominus \mathbb{C}1$ that are invariant under all $\lambda(g_j)$, meaning that every element of Γ_p can be written as a product of elements in $\{g_1,\ldots,g_k\}$. So, we get that $g_1,\ldots,g_k$ generate Γ_p. $\square$

Having proven Lemma 7.8, we are now ready to prove Theorem 7.5.

Proof of Theorem 7.5. As in (7.2), for every finite group G, subset $F \subset G$ and unitary representation $\pi : G \to \mathcal{U}(H)$, we write

$$\pi(F) := \frac{1}{|F|} \sum_{g\in F} \pi(g) \; .$$

For every prime number p, we write $\Gamma_p = \mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$. We construct by induction on n a sequence of prime numbers p_n and a generating set

$$F_n \subset K_n := \prod_{j=1}^{n} \Gamma_{p_j}$$

such that, denoting by $\theta_{n-1} : K_n \to K_{n-1}$ to projection onto the first $n-1$ coordinates, the following properties hold.

1. $\theta_{n-1}(F_n) = F_{n-1}$ and the map $\theta_{n-1} : F_n \to F_{n-1}$ is an r_n-fold covering with $r_n \geq 2$.

2. If $\pi \in \mathrm{Irr}(K_n)$ and π does not factor through θ_{n-1}, then $\|\pi(F_n)\| \leq 1/n$.

3. The elements of F_n satisfy no relation of length $\leq n$.

Assume that $p_1,\ldots,p_{n-1}$ and $F_1,\ldots,F_{n-1}$ have been constructed. We have to construct p_n and F_n. Write $k_1 = |F_{n-1}|$ and put $k_0 = \max\{2n+1, k_1\}$. By Lemma 7.8, we can choose $k_2 > k_0$, a prime number p_n and a subset $F \subset \Gamma_{p_n}$ with $|F| = k_2$ such that the elements of F satisfy no relation of length $\leq 3n$ and such that $\|\pi(F)\| \leq 1/(4n)$ for every nontrivial irreducible representation π of Γ_{p_n}.

Write $F_{n-1} = \{g_1,\ldots,g_{k_1}\}$ and $F = \{h_1,\ldots,h_{k_2}\}$. Note that we have chosen $k_2 > \max\{2n+1, k_1\}$. So we can define the subset $F_n \subset K_{n-1} \times \Gamma_{p_n} = K_n$ given by

$$F_n = \{ (g_i, h_i h_j h_i^{-1}) \mid 1 \leq i \leq k_1 \; , \; 1 \leq j \leq k_2 \; , \; i \neq j \} \; .$$

Note that $\theta_{n-1}(F_n) = F_{n-1}$ and that the map $\theta_{n-1} : F_n \to F_{n-1}$ is a $(k_2 - 1)$-fold covering.

Every irreducible representation $\pi \in \mathrm{Irr}(K_n)$ that does not factor through θ_{n-1} is of the form $\pi = \pi_1 \otimes \pi_2$ with $\pi_1 \in \mathrm{Irr}(K_{n-1})$ and with π_2 being a nontrivial irreducible representation of Γ_{p_n}. Note that

$$\pi(F_n) = \frac{1}{k_1} \sum_{i=1}^{k_1} \bigl(\pi_1(g_i) \otimes \pi_2(h_i)\, T_i\, \pi_2(h_i)^*\bigr) \; ,$$

where

$$T_i := \frac{1}{k_2 - 1} \sum_{1 \leq j \leq k_2 \, , j \neq i} \pi_2(h_j) \; .$$

For every fixed $i \in \{1, \ldots, k_1\}$, we have

$$T_i = \frac{k_2}{k_2 - 1} \pi_2(F) - \frac{1}{k_2 - 1} \pi_2(h_i) \; .$$

Therefore,

$$\|T_i\| < 2\, \|\pi_2(F)\| + \frac{1}{2n} \leq \frac{1}{n} \; . \tag{7.4}$$

It then also follows that $\|\pi(F_n)\| < 1/n$.

We next prove that F_n is a generating set of K_n. Fix $i \in \{1, \ldots, k_1\}$. For all $s, t \in \{1, \ldots, k_2\}$ with $s \neq i$ and $t \neq i$, we have

$$(g_i, h_i h_s h_i^{-1})\, (g_i, h_i h_t h_i^{-1})^{-1} = (e, h_i\, h_s h_t^{-1}\, h_i^{-1}) \; .$$

It thus suffices to prove that the set $H_i := \{h_s h_t^{-1} \mid s, t \in \{1, \ldots, k_2\} \setminus \{i\}\}$ generates Γ_{p_n} for each $i \in \{1, \ldots, k_1\}$.

Denote by λ_0 the regular representation of Γ_{p_n} restricted to $\ell^2(\Gamma_{p_n}) \ominus \mathbb{C}1$. Define

$$R_i = \frac{1}{k_2 - 1} \sum_{1 \leq j \leq k_2 \, , j \neq i} \lambda_0(h_j) \; .$$

By (7.4), we get that $\|R_i\| < 1$. Then also $\|R_i R_i^*\| < 1$. So, there is no non zero function in $\ell^2(\Gamma_{p_n}) \ominus \mathbb{C}1$ that is invariant under all $\lambda(h)$, $h \in H_i$. It follows that each H_i is a generating set of Γ_{p_n}.

Denote by $\eta_n : K_n \to \Gamma_{p_n}$ the projection onto the last coordinate. If the elements of F_n satisfy any relation of length $\leq n$, applying η_n will give a nontrivial relation of length $\leq 3n$ between the elements of F. Since such relations do not exist, we have proved that the elements of F_n satisfy no relation of length $\leq n$.

Define $K = \prod_{n=1}^\infty \Gamma_{p_n}$ and still denote by $\theta_n : K \to K_n$ the projection onto the first n coordinates. Define

$$F = \{k \in K \mid \theta_n(k) \in F_n \;\text{ for all } n \geq 1\} \; .$$

Note that $F \subset K$ is closed and $\theta_n(F) = F_n$. Denoting by $\langle F \rangle$ the subgroup of K generated by F, we get that $\theta_n(\langle F \rangle) = K_n$ for all n. So, $\langle F \rangle$ is dense in K, meaning that F topologically generates K.

Since each map $\theta_{n-1} : F_n \to F_{n-1}$ is an r_n-fold covering, there is a unique probability measure ν_0 on K such that $(\theta_n)_*(\nu_0)$ is the normalized counting measure on F_n for each n. Since $r_n \geq 2$ for all n, the measure ν_0 is non atomic. Note that the support of ν_0 equals F. Define the symmetric probability measure ν on K given by $\nu(\mathcal{U}) = (\nu_0(\mathcal{U}) + \nu_0(\mathcal{U}^{-1}))/2$ for all Borel sets $\mathcal{U} \subset K$. The support of ν equals $F \cup F^{-1}$. Since $\lambda(\nu) = (\lambda(\nu_0) + \lambda(\nu_0)^*)/2$, to conclude the proof of the theorem, it suffices to prove that F is free and that ν_0 is a c_0 probability measure.

Let $g_1^{\varepsilon_1} \cdots g_m^{\varepsilon_m}$ be a reduced word of length m with $g_1, \ldots, g_m \in F$. Take $n \geq m$ large enough such that $\theta_n(g_i) \neq \theta_n(g_{i+1})$ whenever $g_i \neq g_{i+1}$. We then get that $\theta_n(g_1)^{\varepsilon_1} \cdots \theta_n(g_m)^{\varepsilon_m}$ is a reduced word of length $m \leq n$ in the elements of F_n. It follows that

$$e \neq \theta_n(g_1)^{\varepsilon_1} \cdots \theta_n(g_m)^{\varepsilon_m} = \theta_n\big(g_1^{\varepsilon_1} \cdots g_m^{\varepsilon_m}\big) \; .$$

So, $g_1^{\varepsilon_1} \cdots g_m^{\varepsilon_m} \neq e$ and we have proven that F is free.

We finally prove that $\|\pi(\nu_0)\| < 1/m$ for every irreducible representation π of K that does not factor through $\theta_m : K \to K_m$. Since there are only finitely many irreducible representations that do factor through $\theta_m : K \to K_m$, this will conclude the proof of the theorem. Let π be such an irreducible representation. There then exists a unique $n > m$ such that $\pi = \pi_0 \circ \theta_n$ and π_0 is an irreducible representation of K_n that does not factor through $\theta_{n-1} : K_n \to K_{n-1}$. But then $\pi(\nu_0) = \pi_0(F_n)$ and thus

$$\|\pi(\nu_0)\| = \|\pi_0(F_n)\| \leq \frac{1}{n} < \frac{1}{m} \; .$$

□

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